You might find it useful to dig out a first-order linear equation problem you have already solved and read it along with this solution to make sense of what is happening. I will try to mimic as much as possible how you would solve this if you had actual functions $a(x)$, $m(x)$ and $s(x)$ for which you were solving this DE, so you should be able to compare what I am doing to what you did.

On Exam 1, you were asked to solve the differential equation

$$a(x)\,y' - m(x)\,y = s(x) \quad y(x_0) = y_0 \quad x_0 < 0$$

where $a(0) = 0$, and $a(x) \neq 0$ for all other values of x.

The first thing you should do is look at what is in front of y' and make sure that that is 1. To make this happen we divide through by $a(x)$, thus getting:

$$y' - \frac{m(x)}{a(x)}\,y = \frac{s(x)}{a(x)}$$

Now we want our integrating factor. To find it we must first integrate the function $-\frac{m(x)}{a(x)}$. If we had actual functions, we would just do this and get a function $F(x)$ out of our integration. (For example, if $-\frac{m(x)}{a(x)}$ were $\sin x$ we would get $-\cos x$ as our $F(x)$.) What is important to remember from calculus is that $F(x)$ is not just any function, it is a function such that

$$\frac{dF(x)}{dx} = -\frac{m(x)}{a(x)} \tag{1}$$

(If this seems confusing, just think of our example above, $\frac{d}{dx}(-\cos x) = \sin x$, and that's why $-\cos x$ is the anti-derivative of $\sin x$.) Now that we have found $F(x)$, the integrating factor is simply $e^{F(x)}$, and we multiply through to get:

$$e^{F(x)}y' - e^{F(x)}\frac{m(x)}{a(x)}\,y = e^{F(x)}\frac{s(x)}{a(x)}$$

We recognize that the left-hand side of this equation is simply

$$\frac{d}{dx}\left(e^{F(x)}\,y\right)$$

so that

$$\frac{d}{dx}\left(e^{F(x)}\,y\right) = e^{F(x)}\frac{s(x)}{a(x)}$$

and now we want to integrate both sides.
Well, if we had actual functions for $F(x)$, $s(x)$ and $a(x)$, we would just integrate, but just as before we don't so we pick a function $G(x)$ such that

$$\frac{dG(x)}{dx} = e^{F(x)}\frac{s(x)}{a(x)} \tag{2}$$

Before, when we were computing the integrating factor, we only needed one function that had this property, so we just picked $F(x)$ and were happy with ourselves (just like if we had to integrate $\sin x$ and we got $-\cos x$, that's good enough for the integrating factor) but here we need *all* functions that have $e^{F(x)}\frac{s(x)}{a(x)}$ as their derivative. Thankfully, once we have $G(x)$, we know that we can get them all by looking at $G(x)+C$ for different constants C. (Remember, if you were actually solving this with functions and at this point you had to integrate $3x^2$ for example, you would not just write x^3, you would write x^3+C. If you don't believe me, look at your notes, or at the example you are following along.)
And so we get that

$$e^{F(x)}\,y = G(x) + C$$

and

$$y = e^{-F(x)}\,G(x) + C\,e^{-F(x)}.$$

We must now use our initial condition to find what C is. Plugging in x_0, we get:

$$y_0 = e^{-F(x_0)}\,G(x_0) + C\,e^{-F(x_0)}$$

and solving for C gives

$$C = y_0\,e^{F(x_0)} - G(x_0)$$

So morally right now we have our solution, but the question on the exam asked for the answer in terms of $a(x)$, $m(x)$, $s(x)$, y_0, and x_0, so we will manipulate our answer a little bit to get rid (a little bit) of the $F(x)$ and $G(x)$ that we made up. Here we go:

$$\begin{aligned} y &= e^{-F(x)}\,G(x) + \left(y_0\,e^{F(x_0)} - G(x_0)\right)e^{-F(x)} \\ &= e^{-F(x)}(G(x)-G(x_0)) + y_0\,e^{-(F(x)-F(x_0))} \end{aligned}$$

And now comes the trickiest part of the problem. Think of $\int_a^b \sin x\,dx$. The answer is $-\cos b + \cos a$. How did you do this? You found a function ($-\cos x$) that had as its derivative $\sin x$ and you plugged in the upper bound, and then plugged in the lower bound. Now, because of equations (1) and (2) above, we have:

$$\int_{x_0}^{x} -\frac{m(t)}{a(t)}\,dt = F(x) - F(x_0)$$

and

$$\int_{x_0}^{x} e^{F(t)}\frac{s(t)}{a(t)}\,dt = G(x) - G(x_0)$$

(This is where the so-called *dummy variable* t comes in. Hopefully you have thought about this and figured out why I can't use x's inside the integrals here (try writing it with x's maybe you'll see how wrong it is). If you haven't and would not think of using a dummy variable here, please talk to a classmate or a TA about this.) So our solution is

$$y = e^{-F(x)}\left(\int_{x_0}^{x} e^{F(t)}\frac{s(t)}{a(t)}\,dt\right) + y_0\,e^{-\left(\int_{x_0}^{x} -\frac{m(t)}{a(t)}\,dt\right)}$$

or, clearing our double negative sign,

$$y = e^{-F(x)} \left(\int_{x_0}^{x} e^{F(t)} \frac{s(t)}{a(t)}\, dt \right) + y_0\, e^{\left(\int_{x_0}^{x} \frac{m(t)}{a(t)}\, dt \right)}$$

where $F(x)$ is any function such that $\frac{dF(x)}{dx} = -\frac{m(x)}{a(x)}$. (Question for the super-achievers: If I picked a different function $H(x)$ that had the same property that $\frac{dH(x)}{dx} = -\frac{m(x)}{a(x)}$, would my answer be different? As a hint, I can tell you that it wouldn't, otherwise my answer couldn't be the right answer because it would depend on my choice of $F(x)$.)

Now, how is this solution different than the one Professor Smith presented in class and posted on her website? The difference is only one of notation. Professor Smith uses the symbols

$$\int^{x} -\frac{m(t)}{a(t)}\, dt$$

to say, in a much shorter way, "any function that has $-\frac{m(x)}{a(x)}$ as its derivative". To make things clearer, I called such a function $F(x)$ in my argument, but any reasonable way to denote this function is a valid way to solve this problem, and Professor Smith is simply using a notation that you might not be accustomed to. Notice that in Professor Smith's notation, if

$$F(x) = \int^{x} -\frac{m(t)}{a(t)}\, dt$$

then

$$F(x_0) = \int^{x_0} -\frac{m(t)}{a(t)}\, dt$$

so that

$$-F(x_0) = -\int^{x_0} -\frac{m(t)}{a(t)}\, dt = \int_{x_0} -\frac{m(t)}{a(t)}\, dt$$

and

$$F(x) - F(x_0) = \int_{x_0}^{x} -\frac{m(t)}{a(t)}\, dt$$

which is all that really matters in the end.

The region of validity is for $x < 0$. There are many ways to think about this. The first, and maybe most obvious way, is to think back to everything we have done in the course of solving this problem. Everything was pretty safe, except the very first thing we did: we divided. To engage in safe division, we must make sure that the thing we are dividing by is not 0. Since we divided by $a(x)$, this means that x cannot be 0, since $a(x)$ is 0 only when $x = 0$. Now please, please notice that it is not enough to say $x \neq 0$. You *must* instead look at the initial value ($x_0 < 0$) and the pick the side of the singularity (or of the problem if you don't like fancy words) that contains your initial value. So the answer is $x < 0$, since x_0 is on the left of 0.

Another way to think about it is to look at our final answer (which is usually how you would do this if you had actual functions $a(x)$, $m(x)$ and $s(x)$) and look at where there are problems. Everything looks pretty good, except that if $x > 0$, we will integrate from x_0 to x, which will make us go through $t = 0$. There, $a(t)$ is 0, so we have an improper integral, which if MATH 222 went well, you have learned to hate. However, if $x < 0$, then we do not ever need to come close to $t = 0$ and the solution is well-defined. So the solution is valid if $x < 0$.