Solutions for Homework # 3, Problems 3-6
Math 320, Lecture 2
Spring 2009

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3. The Forward Euler method gives us the following formula.

$$\tilde{y}_{n+1} = \tilde{y}_n + hf(x_n, \tilde{y}_n)$$

In this specific problem, we see that $f(x,y) = y + xy^3$. Thus, $f(x_0, y_0) = y_0 + x_0\,(y_0)^3$. Using the above formula, we get

$$\tilde{y}(x_0 + h) = \tilde{y}_1 = y_0 + h(y_0 + x_0\,(y_0)^3).$$

Because the Forward Euler method has error of the order of $\mathcal{O}(h)$, we expect that when h is decreased by a factor of 3, the error will also be decreased by a factor of 3.

4. (a) This problem is basically identical to # 3. In this specific problem, we see that $f(x,y) = 2 + \frac{y}{x}$. Thus, $f(x_0, y_0) = 2 + \frac{y_0}{x_0}$. Using the above formula, we get

$$\tilde{y}(x_0 + h) = \tilde{y}_1 = y_0 + h\left(2 + \frac{y_0}{x_0}\right).$$

(b) The Backward Euler method gives us the following formula.

$$\tilde{y}_{n+1} = \tilde{y}_n + hf(x_{n+1}, \tilde{y}_{n+1})$$

Since $x_1 = x_0 + h$, we see that

$$f(x_1, \tilde{y}_1) = 2 + \frac{\tilde{y}_1}{x_1} = 2 + \frac{\tilde{y}_1}{x_0 + h}.$$

If we plug this into the above formula, we can then solve for $\tilde{y}_1$ and get the following.

$$\begin{aligned}
\tilde{y}_1 &= y_0 + h\left(2 + \frac{\tilde{y}_1}{x_0 + h}\right) \\
&= y_0 + 2h + \frac{h\tilde{y}_1}{x_0 + h} \\
\tilde{y}_1 - \frac{h\tilde{y}_1}{x_0 + h} &= y_0 + 2h \\
\tilde{y}_1\left(1 - \frac{h}{x_0 + h}\right) &= y_0 + 2h \\
\tilde{y}_1 &= \frac{y_0 + 2h}{1 - \frac{h}{x_0 + h}} \\
\tilde{y}_1 &= \frac{(y_0 + 2h)(x_0 + h)}{x_0 + h - h} \\
\tilde{y}_1 &= \frac{(y_0 + 2h)(x_0 + h)}{x_0}
\end{aligned}$$

Thus, $\tilde{y}_1 = \frac{(y_0 + 2h)(x_0 + h)}{x_0}$.

5. (a) Note that the differential equation

$$\frac{dy}{dx} = y - 2xy$$

is separable. So we can use the Separation of Variables technique discussed in class.

$$\begin{aligned}\frac{dy}{dx} &= y - 2xy \\ \frac{1}{y}dy &= (1-2x)dx \\ \int \frac{1}{y}dy &= \int (1-2x)dx \\ \ln|y| &= x - x^2 + C \\ y &= \pm e^{x-x^2+C} \\ &= De^{x-x^2}\end{aligned}$$

If we use the initial value $y(0) = -2$, we can solve for D.

$$\begin{aligned}y &= De^{x-x^2} \\ -2 &= De^{0-0^2} \\ -2 &= D\end{aligned}$$

Thus, by the method of separation of variables, we see that the solution is

$$y = -2e^{x-x^2}$$

and this is valid for all x.

(b) We now wish to write out the Taylor series centered at $a = x_0$. In our case, $x_0 = 0$. Recall the formula for Taylor series expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Since we only need to keep the first four nonzero terms, we only need to take a few derivatives.

$$\begin{aligned}y(x) &= -2e^{x-x^2} \\ y'(x) &= -2(1-2x)e^{x-x^2} \\ y''(x) &= -2(1-2x)^2e^{x-x^2} + 4e^{x-x^2} \\ y'''(x) &= -2(1-2x)^3e^{x-x^2} + (-2(2)(1-2x)(-2)e^{x-x^2}) + 4(1-2x)e^{x-x^2} \\ &= -2(1-2x)^3e^{x-x^2} + 8(1-2x)e^{x-x^2} + 4(1-2x)e^{x-x^2}\end{aligned}$$

By plugging in $a = 0$, we get the following.

$$\begin{aligned}y(0) &= -2e^{0-0^2} = -2 \\ y'(0) &= -2(1-2(0))e^{0-0^2} = -2 \\ y''(0) &= -2(1-2(0))^2e^{0-0^2} + 4e^{0-0^2} = 2 \\ y'''(0) &= -2(1-2(0))^3e^{0-0^2} + 8(1-2(0))e^{0-0^2} + 4(1-2(0))e^{0-0^2} = 10\end{aligned}$$

If we put this together with the formula above we get that

$$\begin{aligned}y(x) &\approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 \\ &\approx -2 - 2x + \frac{2}{2}x^2 + \frac{10}{3!}x^3 \\ &\approx -2 - 2x + x^2 + \frac{5}{3}x^3.\end{aligned}$$

Thus, $y(x_0 + h) \approx -2 - 2h + h^2 + \frac{5}{3}h^3$.

Alternatively, we could have used our previous knowledge of the Taylor series for e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$$

Plugging in $x - x^2$, we get:

$$e^{x-x^2} = 1 + (x - x^2) + \frac{(x - x^2)^2}{2} + \frac{(x - x^2)^3}{6} + ...$$

We can now multiply by -2:

$$-2e^{x-x^2} = -2 - 2x + 2x^2 - (x - x^2)^2 - \frac{(x - x^2)^3}{3} + ...$$

Finally we expand to get

$$-2e^{x-x^2} = -2 - 2x + 2x^2 - x^2 + 2x^3 - x^4 - \frac{x^3}{3} + ... = -2 - 2x + x^2 + \frac{5}{3}x^3 + ...$$

(c) The Forward Euler method gives us the following formula.

$$\tilde{y}_{n+1} = \tilde{y}_n + hf(x_n, \tilde{y}_n)$$

Note that in this problem, $f(x, y) = y - 2xy$, $x_0 = 0$ and $y_0 = -2$. Thus, we get the following approximation for y_1.

$$\begin{aligned}
\tilde{y}(x_0 + h) &= \tilde{y}_1 \\
&= \tilde{y}_0 + hf(x_0, \tilde{y}_0) \\
&= -2 + hf(0, -2) \\
&= -2 + h((-2) - 2(0)(-2)) \\
&= -2 - 2h
\end{aligned}$$

So, $\tilde{y}(x_0 + h) = -2 - h$.

(d) We see that the first two terms of the Taylor series approximation for $y(x_0 + h)$ are the same as the first two terms of $\tilde{y}(x_0 + h)$.

6. (a) Note that the differential equation

$$\frac{dy}{dx} = -\frac{\sin(x)\cos(x)}{y}$$

is separable. So we can use the Separation of Variables technique discussed in class.

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\sin(x)\cos(x)}{y} \\
y\,dy &= -\sin(x)\cos(x)\,dx \\
\int y\,dy &= \int -\sin(x)\cos(x)\,dx \\
y^2 &= \cos^2(x) + C
\end{aligned}$$

If we use the initial value $y(0) = 1$, we can solve for C.

$$\begin{aligned}
y^2 &= \cos^2(x) + C \\
1^2 &= \cos^2(0) + C \\
1 &= 1 + C \\
C &= 0
\end{aligned}$$

Thus, by the method of separation of variables, we see that $y^2 = \cos^2(x)$. So, y must be plus or minus $\cos(x)$.

$$y = \pm\cos(x)$$

When we plug in the initial condition again, we see that we must choose the positive rather than the negative. Thus, the solution is

$$y = \cos(x)$$

and this is valid for all x.

(b) One can determine the Taylor series for this problem in the same way as we did in Problem 5. However, $y = \cos(x)$ is a very common function and we know that

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + ...$$

If we only write out the first four nonzero terms, we see that

$$\cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.$$

(c) The Improved Euler Method gives us the following formula:

$$\tilde{y}_{n+1} = \tilde{y}_n + h\frac{f(x_n, \tilde{y}_n) + f(x_{n+1}, S_n)}{2},$$

where $S_n = \tilde{y}_n + hf(x_n, \tilde{y}_n)$. Note that in this problem, we only wish to find

$$\tilde{y}(x_0 + h) = \tilde{y}_1 = y_0 + h\frac{f(x_0, y_0) + f(x_1, S_0)}{2}.$$

Also note that in this problem, $f(x, y) = -\frac{\sin(x)\cos(x)}{y}$, $x_0 = 0$, and $y_0 = 1$.

$$\begin{aligned} f(x_0, y_0) &= f(0, 1) \\ &= -\frac{\sin(0)\cos(0)}{1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} S_0 &= y_0 + hf(x_0, y_0) \\ &= 1 + h(0) \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(x_1, S_0) &= f(0 + h, 1) \\ &= f(h, 1) \\ &= -\frac{\sin(h)\cos(h)}{1} \\ &= -\sin(h)\cos(h) \\ &= -\frac{1}{2}\sin(2h) \end{aligned}$$

Thus, we get that

$$\begin{aligned} \tilde{y}(x_0 + h) &= y_0 + h\frac{f(x_0, y_0) + f(x_1, S_0)}{2} \\ &= 1 + h\frac{0 + (-\frac{1}{2}\sin(2h))}{2} \\ &= 1 - \frac{1}{4}h\sin(2h). \end{aligned}$$

Note that if this problem required you to write out the Taylor series expansion for the approximation we just found (which it doesn't), you could easily find the Taylor approximation for $\sin(2h)$ by first using the fact that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

and then plugging in $x = 2h$.

(d) Because the Improved Euler method has $\mathcal{O}(h^2)$, we expect that if the step size is decreased from h to $\frac{h}{4}$, the error will decrease by a factor of $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$.