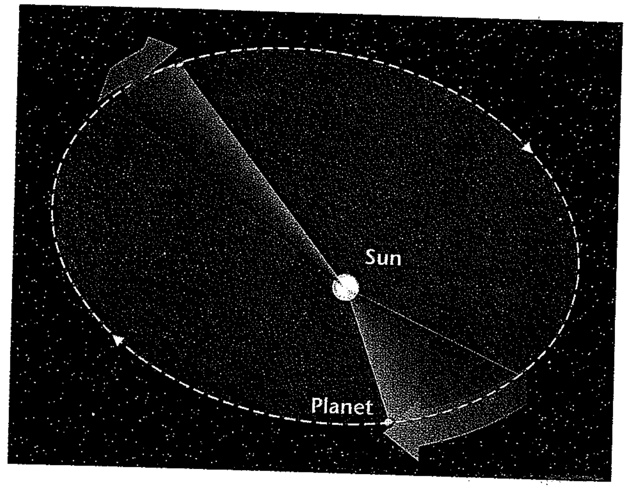
ASTRONOMY

(A) The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cos \theta}$$

where r is measured in miles and the sun is at the pole. Graph the orbit. Use TRACE to find the distance from Mercury to the sun at **aphelion** (greatest distance from the sun) and at **perihelion** (shortest distance from the sun).

(B) Johannes Kepler (1571–1630) showed that a line joining a planet to the sun sweeps out equal areas in space in equal intervals in time (see the figure). Use this information to determine whether a planet travels faster or slower at aphelion than at perihelion. Explain your answer.



8-5 Complex Numbers and De Moivre's Theorem

Utilizing polar concepts studied in Section 8-4, we now show how complex numbers can be written in polar form. A brief review of Section 1-4 on complex numbers should prove helpful before proceeding further.

Rectangular Form

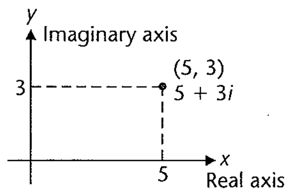
Recall from Section 1-4 that a complex number is any number that can be written in the form

$$x + yi$$

where x and y are real numbers and i is the imaginary unit. (We use $x + yi$ and $x + iy$ interchangeably; each has its advantages in keeping notation simple.) Thus, associated with each complex number $x + yi$ is a unique ordered pair of real numbers (x, y), and vice versa. For example,

$$5 + 3i \qquad \text{corresponds to} \qquad (5, 3)$$

Associating these ordered pairs of real numbers with points in a rectangular coordinate system, we obtain a **complex plane** (Fig. 1). When complex numbers are associated with points in a rectangular coordinate system, we refer to the x axis as the **real axis** and the y axis as the **imaginary axis.** The complex number $x + yi$ is said to be in **rectangular form.**



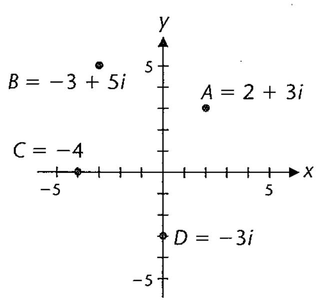
› **Figure 1**
Complex plane.

EXAMPLE 1 Plotting in the Complex Plane

Plot the following complex numbers in a complex plane:

$$A = 2 + 3i \qquad B = -3 + 5i \qquad C = -4 \qquad D = -3i$$

SOLUTION



MATCHED PROBLEM 1

Plot the following complex numbers in a complex plane:

$$A = 4 + 2i \qquad B = 2 - 3i \qquad C = -5 \qquad D = 4i$$

››› EXPLORE-DISCUSS 1

On a *real number line* there is a one-to-one correspondence between the set of real numbers and the set of points on the line: each real number is associated with exactly one point on the line and each point on the line is associated with exactly one real number. Does such a correspondence exist between the set of complex numbers and the set of points in an extended plane? Explain how a one-to-one correspondence can be established.

› Polar Form

Each point (x, y) of the plane corresponds to a unique complex number z, namely, in *rectangular form,* $z = x + iy$. But the point (x, y) can also be specified by polar coordinates. Therefore the complex number z can be given a polar form that depends on r and θ. The **polar form** of z is written $z = re^{i\theta}$. (When convenient, we write $re^{\theta i}$ in place of $re^{i\theta}$.)

	Points	Complex numbers
Rectangular form	(x, y)	$x + iy$
Polar form	(r, θ)	$re^{i\theta}$

The point with rectangular coordinates $(1, 1)$ has polar coordinates $(\sqrt{2}, \pi/4)$. (Why?) Therefore the complex number $z = 1 + i$ has the polar form $z = \sqrt{2}e^{i\pi/4}$. A graphing calculator can convert a complex number in rectangular form to polar form and vice versa (see Fig. 2, where θ is in radians and numbers are displayed to two decimal places).

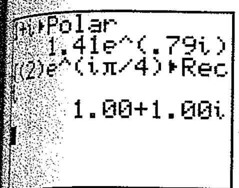
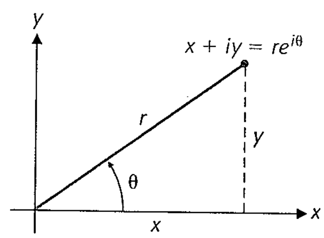


Figure 2
$1 + i = \sqrt{2}e^{i\pi/4}$.

The polar–rectangular relationships of Section 8-4 imply the following connections between the rectangular and polar forms of a complex number.

› POLAR–RECTANGULAR RELATIONSHIPS FOR COMPLEX NUMBERS



If $x + iy = re^{i\theta}$, then

$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$

$y = r\sin\theta \qquad \tan\theta = \dfrac{y}{x}, \qquad x \neq 0$

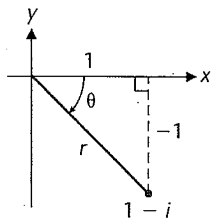
Therefore: $x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and $e^{i\theta} = \cos\theta + i\sin\theta$

If $z = re^{i\theta}$, then the number r is called the **modulus,** or **absolute value,** of z and is denoted by **mod** z or $|z|$. The angle θ (in radians or degrees) is called the **argument** of z and is denoted by **arg** z. Recall that (r, θ) and $(r, \theta + 2\pi)$ represent the same point in polar coordinates. Therefore, $z = re^{i\theta} = re^{i(\theta + 2\pi)}$. So the argument of a complex number is not unique. But we usually choose the argument θ so that $-\pi < \theta \leq \pi$ (or $-180° < \theta \leq 180°$).

EXAMPLE 2 From Rectangular to Polar Form

Write parts A–C in polar form, θ in radians, $-\pi < \theta \le \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

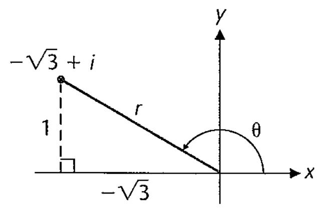
(A) $z_1 = 1 - i$ (B) $z_2 = -\sqrt{3} + i$ (C) $z = -5 - 2i$



› Figure 3

SOLUTIONS

Locate in a complex plane first; then if x and y are associated with special angles, r and θ can often be determined by inspection.



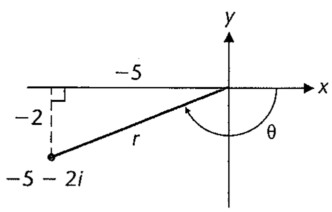
› Figure 4

(A) A sketch shows that z_1 is associated with a special 45° triangle (Fig. 3). Thus, by inspection, $r = \sqrt{2}$, $\theta = -\pi/4$ (not $7\pi/4$), and

$$z_1 = \sqrt{2}e^{i(-\pi/4)}$$

(B) A sketch shows that z_2 is associated with a special 30°–60° triangle (Fig. 4). Thus by inspection, $r = 2$, $\theta = 5\pi/6$, and

$$z_2 = 2e^{i(5\pi/6)}$$



› Figure 5

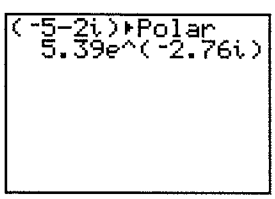
(C) A sketch shows that z_3 is not associated with a special triangle (Fig. 5). So, we proceed as follows:

$$r = \sqrt{(-5)^2 + (-2)^2} = 5.39 \quad \text{To two decimal places}$$
$$\theta = -\pi + \tan^{-1}\left(\tfrac{2}{5}\right) = -2.76 \quad \text{To two decimal places}$$

Thus,

$$z_3 = 5.39e^{(-2.76)i} \quad \text{To two decimal places}$$

Figure 6 shows the same conversion done by a graphing calculator with a built-in conversion routine (with numbers displayed to two decimal places).



› Figure 6
$(-5 - 2i) = 5.39e^{(-2.76)i}$

MATCHED PROBLEM 2

Write parts A–C in polar form, θ in radians, $-\pi < \theta \le \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

(A) $-1 + i$ (B) $1 + i\sqrt{3}$ (C) $-3 - 7i$

EXAMPLE 3 From Polar to Rectangular Form

Write parts A–C in rectangular form. Compute the exact values for parts A and B; for part C, compute a and b for $a + bi$ to two decimal places.

(A) $z_1 = 2e^{(5\pi/6)i}$ (B) $z_2 = 3e^{(-60°)i}$ (C) $z_3 = 7.19e^{(-2.13)i}$

SOLUTIONS

(A) $x + iy = 2e^{(5\pi/6)i}$

$$= 2[\cos(5\pi/6) + i\sin(5\pi/6)] \qquad \cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2},\ \sin\frac{5\pi}{6} = \frac{1}{2}$$

$$= 2\left(\frac{-\sqrt{3}}{2}\right) + i2\left(\frac{1}{2}\right) \qquad \text{Simplify.}$$

$$= -\sqrt{3} + i$$

(B) $x + iy = 3e^{(-60°)i}$

$$= 3[\cos(-60°) + i\sin(-60°)] \qquad \cos(-60°) = \frac{1}{2},\ \sin(-60°) = \frac{-\sqrt{3}}{2}$$

$$= 3\left(\frac{1}{2}\right) + i3\left(\frac{-\sqrt{3}}{2}\right) \qquad \text{Simplify.}$$

$$= \frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

(C) $x + iy = 7.19e^{(-2.13)i}$

$$= 7.19[\cos(-2.13) + i\sin(-2.13)]$$

$$= -3.81 - 6.09\,i$$

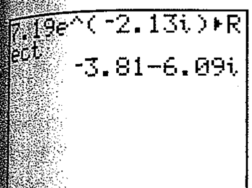


Figure 7
$7.19e^{(-2.13)i} = -3.81 - 6.09i$.

Figure 7 shows the same conversion done by a graphing calculator with a built-in conversion routine.

EXPLORE-DISCUSS 2

If your calculator has a built-in polar-to-rectangular conversion routine, try it on $\sqrt{2}e^{45°i}$ and $\sqrt{2}e^{(\pi/4)i}$, then reverse the process to see if you get back where you started. (For complex numbers in exponential polar form, some calculators require θ to be in radian mode for calculations. Check your user's manual.)

MATCHED PROBLEM 3

Write parts A–C in rectangular form. Compute the exact values for parts A and B; for part C compute a and b for $a + bi$ to two decimal places.

(A) $z_1 = \sqrt{2}e^{(-\pi/2)i}$ (B) $z_2 = 3e^{120°i}$ (C) $z_3 = 6.49e^{(-2.08)i}$

››› EXPLORE-DISCUSS 3

Let $z_1 = \sqrt{3} + i$ and $z_2 = 1 + i\sqrt{3}$.

(A) Find z_1z_2 and z_1/z_2 using the rectangular forms of z_1 and z_2.

(B) Find z_1z_2 and z_1/z_2 using the polar forms of z_1 and z_2, θ in degrees. (Assume the product and quotient exponent laws hold for $e^{i\theta}$.)

(C) Convert the results from part B back to rectangular form and compare with the results in part A.

› Multiplication and Division

There is a particular advantage in representing complex numbers in polar form: multiplication and division become very easy. Theorem 1 provides the reason. (The polar form of a complex number obeys the product and quotient rules for exponents: $b^m b^n = b^{m+n}$ and $b^m/b^n = b^{m-n}$.)

› THEOREM 1 Products and Quotients in Polar Form

If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

1. $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

2. $\dfrac{z_1}{z_2} = \dfrac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

Theorem 1 says that to multiply two complex numbers, you multiply their moduli and add their arguments. Similarly, to divide two complex numbers, you divide their moduli and subtract their arguments.

We establish the multiplication property and leave the quotient property for Problem 64 in Exercises 8-5.

$$
\begin{aligned}
z_1z_2 &= r_1e^{i\theta_1}r_2e^{i\theta_2} && \text{Use polar-rectangular relationships.}\\
&= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) && \text{Multiply.}\\
&= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 && \text{Group real parts and imaginary parts.}\\
&\quad + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\
&= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) && \text{Use sum identities.}\\
&\quad + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]\\
&= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] && \text{Use polar-rectangular relationships.}\\
&= r_1r_2e^{i(\theta_1+\theta_2)}
\end{aligned}
$$

EXAMPLE 4

Products and Quotients

If $z_1 = 8e^{45°i}$ and $z_2 = 2e^{30°i}$, find

(A) z_1z_2 (B) z_1/z_2

SOLUTIONS

(A) $z_1z_2 = 8e^{45°i} \cdot 2e^{30°i}$ $\boxed{= 8 \cdot 2e^{i(45°+30°)}}$* $= 16e^{75°i}$

(B) $\dfrac{z_1}{z_2} = \dfrac{8e^{45°i}}{2e^{30°i}}$ $\boxed{= \frac{8}{2}e^{i(45°-30°)}}$ $= 4e^{15°i}$

MATCHED PROBLEM 4

If $z_1 = 9e^{165°i}$ and $z_2 = 3e^{55°i}$, find

(A) z_1z_2 (B) z_1/z_2

› Powers—De Moivre's Theorem

Abraham De Moivre (1667–1754), of French birth, spent most of his life in London tutoring, writing, and publishing mathematics. He belonged to many prestigious professional societies in England, Germany, and France. He was a close friend of Isaac Newton. The theorem that bears his name gives a formula for raising any complex number to the power n where n is a natural number.

› THEOREM 2 De Moivre's Theorem

If $z = re^{i\theta}$ and n is a natural number, then $z^n = r^ne^{i(n\theta)}$.

De Moivre's theorem follows from the formula for the product of complex numbers in polar form. If $n = 2$ and $z = re^{i\theta}$, then

$$z^2 = re^{i\theta}re^{i\theta} = r^2e^{i(2\theta)}$$

In other words, to square a complex number, you square the modulus and double the argument. Similarly, to cube a complex number you cube the modulus and triple the argument. De Moivre's theorem says that to raise a complex number to the power n, you raise the modulus to the power n and multiply the argument by n.

*Throughout the book, dashed boxes—called think boxes—are used to represent steps that may be performed mentally.

EXAMPLE 5 The Natural Number Power of a Complex Number

Use De Moivre's theorem to find $(1 + i)^{10}$. Write the answer in exact rectangular form.

SOLUTION

First note that the polar form of $1 + i$ is $\sqrt{2}e^{45°i}$. Therefore,

$(1 + i)^{10} = (\sqrt{2}e^{45°i})^{10}$	Use De Moivre's theorem.
$= (\sqrt{2})^{10}e^{(10\cdot 45°)i}$	Simplify.
$= 32e^{450°i}$	Change to rectangular form.
$= 32(\cos 450° + i \sin 450°)$	$\cos 450° = 0$, $\sin 450° = 1$
$= 32(0 + i)$	Simplify.
$= 32i$	Rectangular form

MATCHED PROBLEM 5

Use De Moivre's theorem to find $(1 + i\sqrt{3})^5$. Write the answer in exact polar and rectangular forms.

EXAMPLE 6 The Natural Number Power of a Complex Number

Use De Moivre's theorem to find $(-\sqrt{3} + i)^6$. Write the answer in exact rectangular form.

SOLUTION

First note that the polar form of $\sqrt{3} + i$ is $2e^{150°i}$. Therefore,

$(-\sqrt{3} + i)^6 = (2e^{150°i})^6$	Use De Moivre's theorem.
$= 2^6e^{(6\cdot 150°)i}$	Simplify.
$= 64e^{900°i}$	Change to rectangular form.
$= 64(\cos 900° + i \sin 900°)$	$\cos 900° = -1$, $\sin 900° = 0$
$= 64(-1 + i0)$	Simplify.
$= -64$	Rectangular form

[*Note:* $-\sqrt{3} + i$ must be a sixth root of -64, because $(-\sqrt{3} + i)^6 = -64$.]

MATCHED PROBLEM 6

Use De Moivre's theorem to find $(1 - i\sqrt{3})^4$. Write the answer in exact polar and rectangular forms.

Roots

Let $n > 1$ be an integer. A complex number w is an ***n*th root of *z*** if $w^n = z$. For example, 2 and -2 are square roots (second roots) of 4 because $2^2 = 4$ and $(-2)^2 = 4$. Similarly, $3i$ and $-3i$ are square roots of -9 because $(3i)^2 = -9$ and $(-3i)^2 = -9$. The nth root theorem gives a formula for all of the nth roots of any nonzero complex number.

THEOREM 3 *n*th Root Theorem

Let $n > 1$ be an integer and let $z = re^{i\theta}$ be a nonzero complex number. Then z has n distinct nth roots given by

$$r^{1/n}e^{i(\theta/n + k360°/n)} \qquad k = 0, 1, \ldots, n-1$$

The proof of Theorem 3 is left to Problems 65 and 66 in Exercises 8-5. The nth root theorem implies that every nonzero complex number z has two square roots, three cube roots, four fourth roots, and so on. Furthermore, all n of the nth roots of z have the same modulus, so they all lie on the same circle centered at the origin, and they are equally spaced around that circle.

EXAMPLE 7 Finding All Sixth Roots of a Complex Number

Find six distinct sixth roots of $-1 + i\sqrt{3}$, and plot them in a complex plane.

SOLUTION First write $-1 + i\sqrt{3}$ in polar form:

$$-1 + i\sqrt{3} = 2e^{120°i}$$

Using the nth-root theorem, all six roots are given by

$$2^{1/6}e^{(120°/6+k360°/6)i} = 2^{1/6}e^{(20°+k60°)i} \qquad k = 0, 1, 2, 3, 4, 5$$

Thus,

$$\begin{aligned}
w_1 &= 2^{1/6}e^{(20°+0\cdot60)i} = 2^{1/6}e^{20°i}\\
w_2 &= 2^{1/6}e^{(20°+1\cdot60)i} = 2^{1/6}e^{80°i}\\
w_3 &= 2^{1/6}e^{(20°+2\cdot60)i} = 2^{1/6}e^{140°i}\\
w_4 &= 2^{1/6}e^{(20°+3\cdot60)i} = 2^{1/6}e^{200°i}\\
w_5 &= 2^{1/6}e^{(20°+4\cdot60)i} = 2^{1/6}e^{260°i}\\
w_6 &= 2^{1/6}e^{(20°+5\cdot60)i} = 2^{1/6}e^{320°i}
\end{aligned}$$

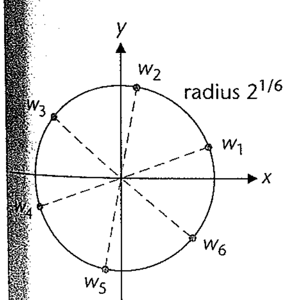


Figure 8

All roots are easily graphed in the complex plane after the first root is located. The root points are equally spaced around a circle of radius $2^{1/6}$ at an angular increment of 60° from one root to the next (Fig. 8).

MATCHED PROBLEM 7

Find five distinct fifth roots of $1 + i$. Leave the answers in polar form and plot them in a complex plane.

EXAMPLE 8 Solving a Cubic Equation

Solve $x^3 + 1 = 0$. Write final answers in rectangular form, and plot them in a complex plane.

SOLUTION

$$x^3 + 1 = 0$$
$$x^3 = -1$$

We see that x is a cube root of -1, and there are a total of three roots. To find the three roots, we first write -1 in polar form:

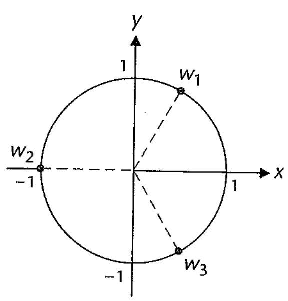
$$-1 = 1e^{180°i}$$

Using the nth-root theorem, all three cube roots of -1 are given by

$$1^{1/3}e^{(180°/3+k360°/3)i} = 1e^{(60°+k120°)i} \qquad k = 0, 1, 2$$

Thus,

$$w_1 = 1e^{60°i} = \cos 60° + i \sin 60° = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$w_2 = 1e^{180°i} = \cos 180° + i \sin 180° = -1$$
$$w_3 = 1e^{300°i} = \cos 300° + i \sin 300° = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$



› Figure 9

(*Note:* This problem can also be solved using factoring and the quadratic formula—try it.)

The three roots are graphed in Figure 9.

MATCHED PROBLEM 8

Solve $x^3 - 1 = 0$. Write final answers in rectangular form, and plot them in a complex plane.

Historical Note

There is hardly an area in mathematics that does not have some imprint of the famous Swiss mathematician Leonhard Euler (1707–1783), who spent most of his productive life at the New St. Petersburg Academy in Russia and the Prussian Academy in Berlin. One of the most prolific writers in the history of the subject, he is credited with making the following familiar notations standard:

$f(x)$	function notation
e	natural logarithmic base
i	imaginary unit, $\sqrt{-1}$

For our immediate interest, he is also responsible for the extraordinary relationship

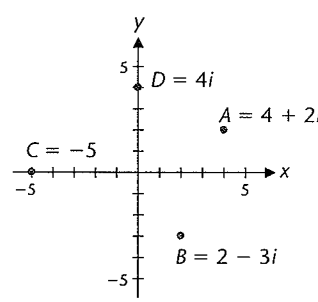
$$e^{i\theta} = \cos\theta + i\sin\theta$$

If we let $\theta = \pi$, we obtain an equation that relates five of the most important numbers in the history of mathematics:

$$e^{i\pi} + 1 = 0$$

ANSWERS TO MATCHED PROBLEMS

1.



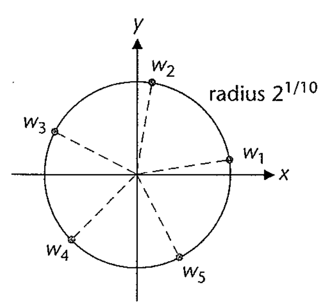
2. (A) $\sqrt{2}e^{i(3\pi/4)}$
(B) $2e^{i(\pi/3)}$
(C) $7.62e^{(-1.98)i}$

3. (A) $-i\sqrt{2}$ (B) $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ (C) $-3.16 - 5.67i$

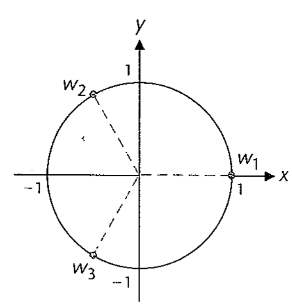
4. (A) $z_1z_2 = 27e^{220°i}$ (B) $z_1/z_2 = 3e^{110°i}$

5. $32e^{300°i} = 16 - i16\sqrt{3}$ **6.** $16e^{(-240°)i} = -8 + i8\sqrt{3}$

7. $w_1 = 2^{1/10}e^{9°i}$, $w_2 = 2^{1/10}e^{81°i}$,
$w_3 = 2^{1/10}e^{153°i}$, $w_4 = 2^{1/10}e^{225°i}$, $w_5 = 2^{1/10}e^{297°i}$



8. $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$



8-5 Exercises

In Problems 1–8, plot each set of complex numbers in a complex plane.

1. $A = 3 + 4i, B = -2 - i, C = 2i$

2. $A = 4 + i, B = -3 + 2i, C = -3i$

3. $A = 3 - 3i, B = 4, C = -2 + 3i$

4. $A = -3, B = -2 - i, C = 4 + 4i$

5. $A = 2e^{(\pi/3)i}, B = \sqrt{2}e^{(\pi/4)i}, C = 4e^{(\pi/2)i}$

6. $A = 2e^{(\pi/6)i}, B = 4e^{\pi i}, C = \sqrt{2}e^{(3\pi/4)i}$

7. $A = 4e^{(-150°)i}, B = 3e^{20°i}, C = 5e^{(-90°)i}$

8. $A = 2e^{150°i}, B = 3e^{(-50°)i}, C = 4e^{75°i}$

In Problems 9–12, convert to the polar form $re^{i\theta}$. For Problems 9 and 10, choose θ in degrees, $-180° < \theta \le 180°$; for Problems 11 and 12 choose θ in radians, $-\pi < \theta \le \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

9. (A) $\sqrt{3} + i$ (B) $-1 - i$ (C) $5 - 6i$

10. (A) $-1 + i\sqrt{3}$ (B) $-3i$ (C) $-7 - 4i$

11. (A) $-i\sqrt{3}$ (B) $-\sqrt{3} - i$ (C) $-8 + 5i$

12. (A) $\sqrt{3} - i$ (B) $-2 + 2i$ (C) $6 - 5i$

In Problems 13–16, change parts A–C to rectangular form. Compute the exact values for parts A and B; for part C compute a and b for $a + bi$ to two decimal places.

13. (A) $2e^{(\pi/3)i}$ (B) $\sqrt{2}e^{(-45°)i}$ (C) $3.08e^{2.44i}$

14. (A) $2e^{30°i}$ (B) $\sqrt{2}e^{(-3\pi/4)i}$ (C) $5.71e^{(-0.48)i}$

15. (A) $6e^{(\pi/6)i}$ (B) $\sqrt{7}e^{(-90°)i}$ (C) $4.09e^{(-122.88°)i}$

16. (A) $\sqrt{3}e^{(-\pi/2)i}$ (B) $\sqrt{2}e^{135°i}$ (C) $6.83e^{(-108.82°)i}$

In Problems 17–22, find z_1z_2 and z_1/z_2 in the polar form $re^{i\theta}$.

17. $z_1 = 7e^{82°i}, z_2 = 2e^{31°i}$

18. $z_1 = 6e^{132°i}, z_2 = 3e^{93°i}$

19. $z_1 = 5e^{52°i}, z_2 = 2e^{83°i}$

20. $z_1 = 3e^{67°i}, z_2 = 2e^{97°i}$

21. $z_1 = 3.05e^{1.76i}, z_2 = 11.94e^{2.59i}$

22. $z_1 = 7.11e^{0.79i}, z_2 = 2.66e^{1.07i}$

In Problems 23–28, use De Moivre's theorem to evaluate each. Leave answers in polar form.

23. $(2e^{30°i})^3$

24. $(5e^{15°i})^3$

25. $(\sqrt{2}e^{10°i})^6$

26. $(\sqrt{2}e^{15°i})^8$

27. $(1 + i\sqrt{3})^3$

28. $(\sqrt{3} + i)^8$

In Problems 29–34, find the value of each expression and write the final answer in exact rectangular form. (Verify the results in Problems 29–34 by evaluating each directly on a calculator.)

29. $(-\sqrt{3} - i)^4$

30. $(-1 + i)^4$

31. $(1 - i)^8$

32. $(-\sqrt{3} + i)^5$

33. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

34. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

For n and z as indicated in Problems 35–40, find all nth roots of z. Leave answers in the polar form $re^{i\theta}$.

35. $z = 8e^{30°i}, n = 3$

36. $z = 8e^{45°i}, n = 3$

37. $z = 81e^{60°i}, n = 4$

38. $z = 16e^{90°i}, n = 4$

39. $z = 1 - i, n = 5$

40. $z = -1 + i, n = 3$

For n and z as indicated in Problems 41–46, find all nth roots of z. Write answers in the polar form $re^{i\theta}$ and plot in a complex plane.

41. $z = 8, n = 3$

42. $z = 1, n = 4$

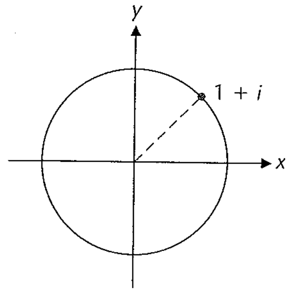
43. $z = -16, n = 4$

44. $z = -8, n = 3$

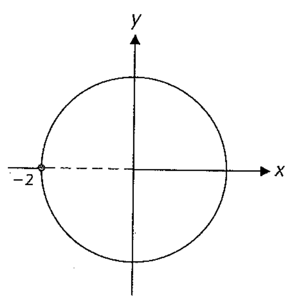
45. $z = i, n = 6$

46. $z = -i, n = 5$

47. (A) Show that $1 + i$ is a root of $x^4 + 4 = 0$. How many other roots does the equation have?
(B) The root $1 + i$ is located on a circle of radius $\sqrt{2}$ in the complex plane as indicated in the figure. Locate the other three roots of $x^4 + 4 = 0$ on the figure and explain geometrically how you found their location.
(C) Verify that each complex number found in part B is a root of $x^4 + 4 = 0$.



48. (A) Show that -2 is a root of $x^3 + 8 = 0$. How many other roots does the equation have?
(B) The root -2 is located on a circle of radius 2 in the complex plane as indicated in the figure. Locate the other two roots of $x^3 + 8 = 0$ on the figure and explain geometrically how you found their location.
(C) Verify that each complex number found in part B is a root of $x^3 + 8 = 0$.



In Problems 49–52, solve each equation for all roots. Write final answers in the polar form $re^{i\theta}$ and exact rectangular form.

49. $x^3 + 64 = 0$

50. $x^3 - 64 = 0$

51. $x^3 - 27 = 0$

52. $x^3 + 27 = 0$

In Problems 53–62, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

53. If two numbers lie on the real axis, then their product lies on the real axis.

54. If two numbers lie on the imaginary axis, then their quotient lies on the imaginary axis.

55. If z is a positive real number, then all of the fourth roots of z are real.

56. If z is a positive real number, then all of the square roots of z are real.

57. If w is a square root of 1, then w is a sixth root of 1.

58. If w is a sixth root of 1, then w is a square root of 1.

59. If w is both a cube root and a fourth root of a nonzero complex number z, then $|w| = 1$.

60. If w is both a cube root and a sixth root of a nonzero complex number z, then $|w| = 1$.

61. If w is both a cube root and a sixth root of a nonzero complex number z, then $w = 1$.

62. If w is both a cube root and a fourth root of a nonzero complex number z, then $w = 1$.

63. Suppose that z is a complex number that is not real. Explain why none of the nth roots of z lies on the x axis.

64. Prove

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

65. Show that

$$[r^{1/n} e^{(\theta/n + k360°/n)i}]^n = re^{i\theta}$$

for any natural number n and any integer k.

66. Show that

$$r^{1/n} e^{(\theta/n + k360°/n)i}$$

is the same number for $k = 0$ and $k = n$.

In Problems 67–70, write answers in the polar form $re^{i\theta}$.

67. Find all complex zeros for $P(x) = x^5 - 32$.

68. Find all complex zeros for $P(x) = x^6 + 1$.

69. Solve $x^5 + 1 = 0$ in the set of complex numbers.

70. Solve $x^3 - i = 0$ in the set of complex numbers.

In Problems 71 and 72, write answers using exact rectangular forms.

71. Write $P(x) = x^6 + 64$ as a product of linear factors.

72. Write $P(x) = x^6 - 1$ as a product of linear factors.