MATH 213

Telescoping Series

Recall:

An infinite sum, which we write
$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \dots + a_n + \dots,$$
is called an infinite series.

We can truncate an infinite series at a finite number n:
$$S_n = a_1 + a_2 + \dots + a_n = \sum_{i=1}^{n} a_i$$
We call S_n the nth partial sum for the series.

We say that the series converges to the number L if
$$\lim_{n\to\infty} S_n = L.$$

If $\lim_{n\to\infty} S_n$ does not exist, we say that the series diverges.

Let's do two examples to practice the vocab:

Example 1:

Consider the series

$$\sum_{i=1}^{\infty} \frac{5}{2}\left(\frac{1}{2}\right)^i$$

we write out the first few terms:

$$\sum_{i=1}^{\infty} \frac{5}{2}\left(\frac{1}{2}\right)^i = \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \ldots$$

we recognize this as a geometric series

with $a = a_1 = \frac{5}{4}$

$$r = \frac{a_2}{a_1} = \frac{5/8}{5/4} = \frac{5}{8} \cdot \frac{4}{5} = \frac{1}{2}$$

If we rewrite the a_i's, we get

$$a_i = \frac{5}{2}\left(\frac{1}{2}\right)^i = \frac{5}{2} \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{i-1} = \frac{5}{4}\left(\frac{1}{2}\right)^{i-1}$$

we now rewrite the sum, letting $j = i-1$

$$\sum_{i=1}^{\infty} \frac{5}{4}\left(\frac{1}{2}\right)^{i-1} = \sum_{j=0}^{\infty} \frac{5}{4}\left(\frac{1}{2}\right)^j$$

From Section 12.1 we know that the nth partial sum is

$$S_n = \sum_{j=0}^{n-1} \frac{5}{4}\left(\frac{1}{2}\right)^j = \frac{5/4\left(\left(1/2\right)^n - 1\right)}{1/2 - 1}$$

we compute

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{5}{4} \frac{\left(\left(1/2\right)^n - 1\right)}{-1/2}$$

$$= \frac{5}{4} \frac{(0-1)}{-1/2} = \frac{5}{4} \cdot 2 = \frac{5}{2}$$

Since the limit exists we say that the series converges to $5/2$ OR

$$\sum_{i=1}^{\infty} \frac{5}{2}\left(\frac{1}{2}\right)^i = \frac{5}{2}$$

Example 2 (You are not responsible for this example)

Consider the series

$$\sum_{i=1}^{\infty} i = 1+2+3+4+\ldots$$

You can look up that the nth partial sum is

$$S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We compute

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2}$$
$$= \lim_{n\to\infty} \frac{n^2+n}{2} = +\infty$$

Since the limit doesnot exist we say that the series diverges.

Now onto telescoping sums.

A telescoping sum is any series in which enough cancellation occurs to let you compute the nth partial sum. I will work out 2 examples:

Example 1:

Consider the series

$$\sum_{i=1}^{\infty} \frac{1}{(i+2)(i+3)} = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$$

Note that this is <u>not</u> a geometric series

since
$$\frac{a_2}{a_1} = \frac{1/20}{1/12} = \frac{12}{20} = \frac{3}{5}$$
$$\frac{a_3}{a_2} = \frac{1/30}{1/20} = \frac{20}{30} = \frac{2}{3}$$
not equal!

But,.. we can rewrite a_i to get some cancellation:

$$a_i = \frac{1}{(i+2)(i+3)} = \frac{1}{i+2} - \frac{1}{i+3}$$

Let's look at the nth partial sum:

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{i+2} - \frac{1}{i+3} \right)$$

$$= \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) +$$

$$+ \dots + \underbrace{\left(\frac{1}{n+1} - \frac{1}{n+2} \right)}_{a_{n-1}} + \underbrace{\left(\frac{1}{n+2} - \frac{1}{n+3} \right)}_{a_n}$$

we are allowed to move the parentheses around to get

$$S_n = \frac{1}{3} + \left(-\frac{1}{4} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{5}\right) + \dots + \left(-\frac{1}{n+2} + \frac{1}{n+2}\right) - \frac{1}{n+3}$$

$$= \frac{1}{3} - \frac{1}{n+3}$$

We now compute

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{n+3}\right) = \frac{1}{3}$$

So the series converges to $1/3$.

Example 2

Consider the series

$$\sum_{i=1}^{\infty}\left(\frac{1}{i^2} - \frac{1}{(i+1)^2}\right)$$

Then the nth partial sum is

$$S_n = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \dots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= 1 - \frac{1}{(n+1)^2}$$

(I went a little fast there, so make sure to work this out carefully if it's not obvious to you how I got $S_n = 1 - \frac{1}{(n+1)^2}$!)

We compute

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)^2}\right) = 1$$

So the series converges to 1.

To master this, answer the following questions:

For each series, find i) nth partial sum

ii) the number to which the series converges.

a) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$

b) $\displaystyle\sum_{i=1}^{\infty} \left[\frac{1}{(i+2)^2} - \frac{1}{(i+3)^2}\right]$

c) $\displaystyle\sum_{i=2}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i}\right)$