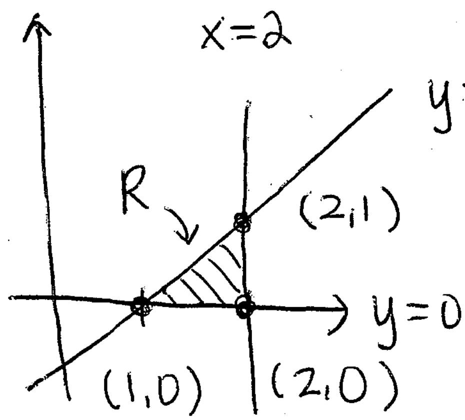
MATH 213

Practicing double integrals (9.6)

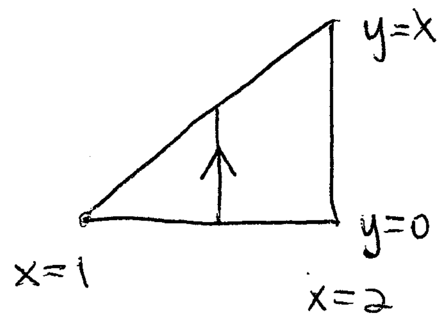
#54 $\iint_R \frac{dy\,dx}{x}$, R bounded by

$1 \le x \le 2$

$0 \le y \le x-1$



1st way: dydx



$$\int_1^2 \int_0^{x-1} \frac{1}{x}\,dy\,dx$$

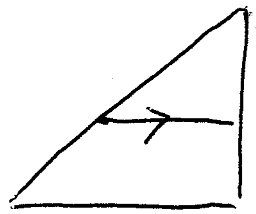
$$= \int_1^2 \frac{1}{x}\, y \Big|_{y=0}^{y=x-1} dx$$

$$= \int_1^2 \frac{1}{x}((x-1)-0)\,dx$$

$$= \int_1^2 \frac{x-1}{x}\,dx = \int_1^2 \left(1-\frac{1}{x}\right)dx$$

$$= (x - \ln|x|\,\Big|_{x=1}^{x=2} = 2 - \ln 2 - 1$$

2nd way: dx dy



$$\int_0^1 \int_{y+1}^{2} \frac{1}{x}\,dx\,dy$$

$$= \int_0^1 \ln|x| \Big|_{x=y+1}^{x=2} dy$$

$$= \int_0^1 (\ln(2) - \ln(y+1))\,dy$$

$$= \ln 2 \cdot y \Big|_{y=0}^{y=1} - \int_0^1 \ln(y+1)\,dy$$

integration by parts!

$u = \ln(y+1)$ $\quad dv = dy$

$du = \frac{1}{y+1}dy$ $\quad v = y$

$$= \ln 2 - \left(y\ln(y+1) \Big|_{y=0}^{y=1} - \int_0^1 \frac{y}{y+1}dy \right)$$

$$= \ln 2 - (\ln 2 - 0) + \int_0^1 \frac{y}{y+1}dy$$

u-substitution!

$u = y+1$ $\quad du = dy$

$y = u-1$

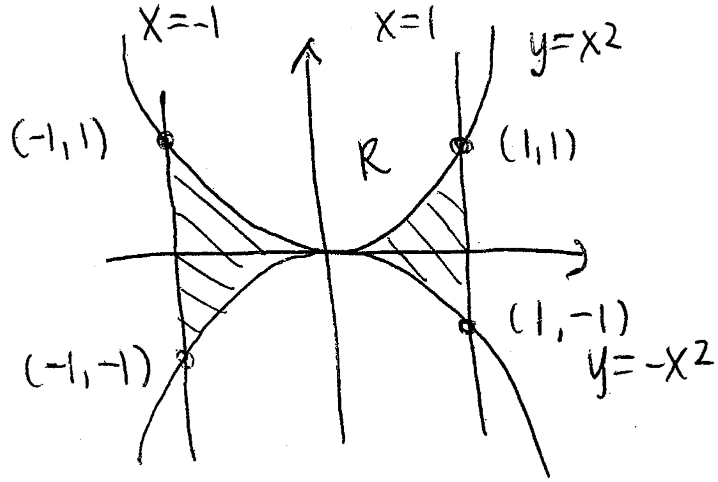
when $y=0$, $u=1$

$y=1$, $u=2$

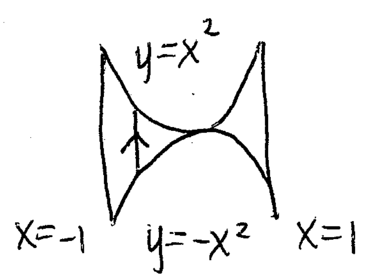
$$= \int_1^2 \frac{u-1}{u}du = \int_1^2 \left(1 - \frac{1}{u}\right)du = u - \ln|u| \Big|_1^2 = 2 - \ln 2 - 1$$

#56 $\iint_R (x^2-y)\,dy\,dx$ R bounded by

$-1 \le x \le 1,\ -x^2 \le y \le x^2$



1st way: dydx



$$\int_{-1}^{1}\int_{-x^2}^{x^2} (x^2-y)\,dy\,dx$$

$$= \int_{-1}^{1} \left(x^2 y - \frac{y^2}{2} \Big|_{y=-x^2}^{y=x^2} \right) dx$$

$$= \int_{-1}^{1} \left(\left(x^4 - \frac{x^4}{2}\right) - \left(-x^4 - \frac{x^4}{2}\right) \right) dx$$

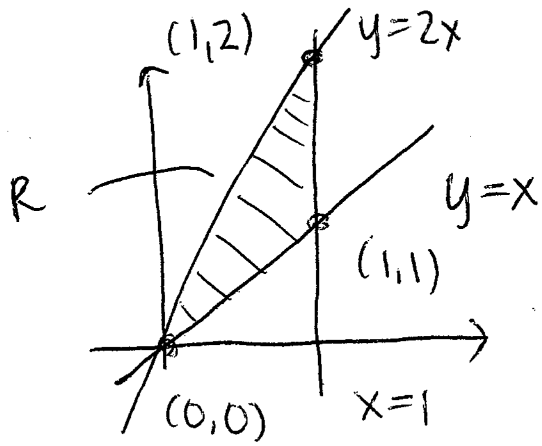
$$= \int_{-1}^{1} \left(x^4 - \frac{x^4}{2} + x^4 + \frac{x^4}{2} \right) dx$$

$$= \int_{-1}^{1} 2x^4\,dx = 2\frac{x^5}{5}\Big|_{x=-1}^{x=1} = \frac{2}{5}\left(1^5 - (-1)^5\right)$$

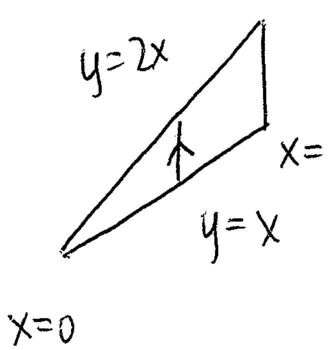
$$= \frac{2}{5}(1+1) = \frac{4}{5}$$

Doing it the second way involves 4 different integrals and about 4 pages of work. It's not for the faint of heart!

#58 $\iint_R x^2y^2\,dx\,dy$ R bounded by $y=x,\ y=2x,\ x=1$



1st way dydx



$$\int_0^1\int_x^{2x} x^2y^2\,dy\,dx$$

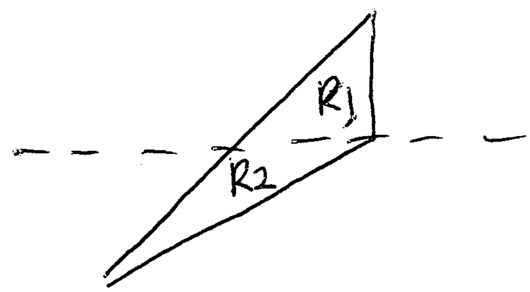
$$=\int_0^1 \left(x^2\frac{y^3}{3}\Big|_{y=x}^{y=2x}\right)dx$$

$$=\int_0^1 \frac{x^2}{3}(8x^3-x^3)\,dx$$

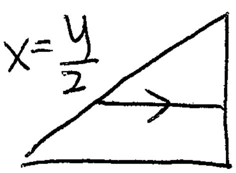
$$=\int_0^1 \frac{7x^5}{3}\,dx=\frac{7}{3}\frac{x^6}{6}\Big|_0^1=\frac{7}{18}$$

2 way $dx\,dy$

we must separate the region R into 2 regions: R_1 and R_2:



R_1: $x = \frac{y}{2}$



$y=2$
$x=1$
$y=1$

$$\int_1^2 \int_{y/2}^1 x^2 y^2 \, dx\, dy$$

$$= \int_1^2 \left(y^2 \frac{x^3}{3} \Big|_{x=y/2}^{x=1} \right) dy$$

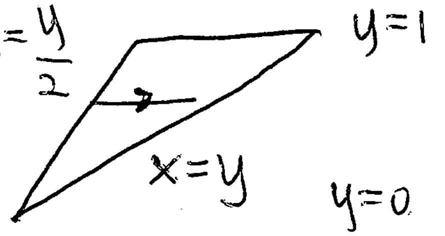
$$= \int_1^2 \frac{y^2}{3}\left(1 - \frac{y^3}{8}\right) dy$$

$$= \int_1^2 \left(\frac{y^2}{3} - \frac{y^5}{24} \right) dy = \frac{y^3}{9} - \frac{y^6}{6 \cdot 24} \Big|_1^2$$

$$= \left(\frac{8}{9} - \frac{2^6}{6 \cdot 24} \right) - \left(\frac{1}{9} - \frac{1}{6 \cdot 24} \right)$$

$$= \frac{7}{9} + \frac{1 - 2^6}{6 \cdot 24}$$

⑥

R_2: 

$$\int_0^1 \int_{y/2}^{y} x^2 y^2 \, dx\, dy$$

$$= \int_0^1 y^2 \frac{x^3}{3} \Big|_{x=y/2}^{x=y} dy = \int_0^1 \frac{y^2}{3}\left(y^3 - \frac{y^3}{8}\right) dy$$

$$= \int_0^1 \frac{7}{24} y^5 \, dy = \frac{7}{24} \frac{y^6}{6} \Big|_0^1 = \frac{7}{6\cdot 24}$$

Now $\iint_R x^2y^2\,dx\,dy = \iint_{R_1} x^2y^2\,dx\,dy + \iint_{R_2} x^2y^2\,dx\,dy$

$$= \frac{7}{9} + \frac{1}{6\cdot 24} - \frac{2^6}{6\cdot 24} + \frac{7}{6\cdot 24}$$

$6\cdot 24 = 2\cdot 3\cdot 8\cdot 3$
$= 24\cdot 9$

$$= \frac{7}{9} + \frac{8}{6\cdot 24} - \frac{2^6}{6\cdot 24}$$

$$= \frac{7}{9} + \frac{1}{18} - \frac{4}{9}$$

$$= \frac{3}{9} + \frac{1}{18}$$

$$= \frac{6}{18} + \frac{1}{18} = \frac{7}{18}$$