Eventown and Oddtown

Calum Buchanan

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1 Introduction

The Eventown and Oddtown problems are problems in extremal combinatorics which concern maximizing the number of subsets of a finite set which are subject to certain parity-related constraints. The problems were solved by Elwyn Berlekamp [1] in 1969. Berlekamp's proofs are entirely algebraic. While Fedor Petrov [2] has a combinatorial proof of the Oddtown theorem, presented here, we have not been able to find a combinatorial proof of the Eventown theorem and pose this question to the reader. We present a proof of the Eventown theorem using linear algebra, as well as a combinatorial proof of the Oddtown theorem. As a side note, this latter proof may in fact be generalized to prove the linear dependence of any $n+1$ vectors in any n-dimensional vector space.

2 Eventown

We will first discuss Eventown. In this peculiar town, all n residents love to form clubs. Not being very particular about their club tastes, the residents just want to maximize the number of clubs, m (even if this means an empty club!). However, there are traditions in the forming of clubs which must be observed:

(i) Each club must have an even number of members;

(ii) No two clubs have exactly the same members;

(iii) Any two clubs must share an even number of members.

Of course, in attempting to maximize m, the residents would like to formalize this problem. They do so as follows.

Let $[n]$ denote the set $\{1, \ldots, n\}$. A club system is a collection of distinct subsets of $[n]$, $\mathcal{C} = \{C_1, \ldots, C_m\}$. Maximizing the number of clubs in Eventown is then equivalent to maximizing the number of distinct sets in $\mathcal{C}$ such that $A \cap B$ is even for all $A, B \in \mathcal{C}$

By pairing up the residents and then taking all possible distinct collections of these pairs, the residents of Eventown find a way to form $2^{n/2}$ clubs. But they wonder if they might be able to form more if only somebody were to come up with a clever strategy. They hire a consultant who returns with bad news.

Theorem 1. *There can be no more than $2^{n/2}$ clubs in Eventown.*

Proof (using linear algebra). Let $\mathcal{C} = \{C_1, \dots, C_m\} \subset 2^{[n]}$ be such that $|C_i \cap C_j|$ is even for every $i, j \in [m]$. It is to be shown that $|\mathcal{C}| \leq 2^{n/2}$.

Let $v_1, \dots, v_m$ denote the incidence vectors of the clubs. That is, $v_i \in \mathbf{F}_2^n$ has entry $j = 1$ if $j \in C_i$ and 0 otherwise. Since the intersection of any two clubs is even, $v_i \cdot v_j = 0$ for all $i, j \in [m]$, where the dot product is taken over $\mathbf{F}_2$. Consider the span V of $\{v_1, \dots, v_m\}$. We note that V must be a subspace of $V^\perp := \{w \in \mathbf{F}_2^n : v \cdot w = 0 \text{ for all } v \in V\}$ for if $u = a_1 v_1 + \cdots + a_m v_m$ and $v = b_1 v_1 + \cdots + b_m v_m$ both lie in V, then each $a_i b_j v_i v_j = 0$, and thus $u \cdot v = 0$.

Now we recall that, when V is a subspace of W, then $\dim V + \dim V^\perp = \dim W$. Since $V \subset \mathbf{F}_2^n$ of dimension n and $V \subset V^\perp$, we have that

$$2 \dim V \leq \dim V + \dim V^\perp = n$$

in which case

$$\dim V \leq n/2$$

which yields the desired result that

$$|V| \leq 2^{n/2}.$$

□

3 Oddtown

Another strange town named Oddtown lies adjacent to Eventown and has similar traditions. The N residents of Oddtown also love forming clubs and would like to maximize the number of them. However, the club-forming traditions in Oddtown differ in one important way from those in Eventown: the number of people in each club must be *odd*, while the number of people shared by any two clubs remains even.

Many of the Oddtown residents are envious of their neighbors as the Eventown residents' pairing strategy always seems to produce more clubs than they can. In fact, the Oddtown residents are quite convinced that they can form no more clubs than the number of residents, N. But how to prove this? There are certainly many ways of forming N clubs in Oddtown. The residents can each form their own, rather boring, single-person club. On the other hand, when the number of residents is even, they can form N lively clubs, each of size $N - 1$.

Theorem 2 (Elwyn Berlekamp, 1969)**.** *An Oddtown with N residents may form no more than N clubs.*

We present a combinatorial proof of this result due to Fedor Petrov [2] as opposed to the algebraic one. This proof is not only intuitive, but versatile as it may be generalized to prove the dependence of a collection of $n + 1$ vectors in any n-dimensional vector space.

Proof. Suppose, for the sake of contradiction, there exists an Oddtown club system $\mathcal{F}$ consisting of at least $N+1$ clubs. Then there are at least 2^{N+1} subcollections of $\mathcal{F}$. For each subcollection A, consider the residents from $\{1,\ldots,N\}$ which are contained in an odd number of clubs in A. By the pigeonhole principle, two of these subcollections, say A and B, must have the same set of residents in an odd number of clubs.

This means that each resident in the subcollection $C := A \triangle B$, i.e. the collection of sets which are in A or B but not both, is contained in an even number of clubs. Let $C = \{U_1,\ldots,U_k\}$ and observe that

$$\sum_{2}^{k} |U_1 \cap U_i| = \sum_{x\in U_1} \sum_{2}^{k} |\{x\} \cap U_i| \equiv |U_1| \mod 2.$$

But now we notice that, if $|U_1|$ is odd like every club in Oddtown, then at least one $|U_1 \cap U_i|$ must also be odd, which is a contradiction. □

4 Further Remarks

Question. *Can you think of a combinatorial proof of the Eventown result?*

References

[1] ER Berlekamp. On subsets with intersections of even cardinality. *Canadian Mathematical Bulletin*, 12(4):471–474, 1969.

[2] Fedor Petrov (https://mathoverflow.net/users/4312/fedor petrov). List of counting proofs instead of linear algebra method in combinatorics. MathOverflow. URL:https://mathoverflow.net/q/230906 (version: 2016-02-12).