Calum's notes on Integration by Substitution

For starters, keep in mind the *chain rule* for the derivative of a composition of functions:

$$\tfrac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Most of the time, integration by substitution is used to "reverse the chain rule." Other times, it just helps us to simplify an equation.

Definition. We will say that two functions $f(x)$ and $g(x)$ are *related* if there is some constant K so that

$$f(x) = K \cdot g(x)$$

(or vice-versa).

Example. $f(x) = 2x^2$ and $g(x) = 6x^2$ are two related functions.

When asked to take an integral which is not immediately apparent, we can ask ourselves:

"is one function inside the integral related to the derivative of another?"

or

"could I simplify this problem if I substituted one function for another?"

If the answer is yes, try using **substitution** to simplify the integral.

Example (Reversing the chain rule)**.** Say we are asked to evaluate an integral of the form

$$\int f(g(x))h(x)\ dx$$

and we know that the *derivative of* $g(x)$ *is related to* $h(x)$, that is

$$\tfrac{d}{dx}(g(x)) = g'(x) = K \cdot h(x)$$

for some constant K.

If we set $u = g(x)$, then we see that $\frac{du}{dx} = g'(x) = K \cdot h(x)$. If we "multiply" by dx on each side (slight abuse of terminology here) and divide by K, then we have

$$\tfrac{1}{K}du = h(x)dx.$$

We may then substitute u for $g(x)$ and $\frac{1}{K}du$ for $h(x)dx$ in our original problem to simplify $\int f(g(x))h(x)\ dx$ to

$$\int f(u) \cdot \tfrac{1}{K}\ du = \tfrac{1}{K} \int f(u)\ du.$$

If the integral of $f(u)$ is easy to solve (and for our purposes it will be), then we win! We have reversed the chain rule. Plug $g(x) = u$ back into the answer you obtain to solve the integral.

Example (Other funky integrals)**.** Say we are asked to evaluate

$$\int x\sqrt{x+3}\ dx.$$

Then we might like to swap some things around to simplify this square root. Try

$$\begin{aligned} u &= x + 3 \\ \tfrac{du}{dx} &= 1 \implies du = dx. \end{aligned}$$

Then $x = u - 3$, so $\int x\sqrt{x+3}\ dx$ becomes

$$\begin{aligned} \int (u-3)\sqrt{u}\ du &= \int u\sqrt{u} - 3\sqrt{u}\ du \\ &= \int u^{3/2}\ du - 3\int u^{1/2}\ du \\ &= \tfrac{2}{5}u^{5/2} - 3(\tfrac{2}{3}u^{3/2}) + C. \end{aligned}$$

Plugging back $u = x+3$, simplifying, and changing fractional exponents back to roots, we obtain the final answer:

$$\int x\sqrt{x+3}\ dx = \tfrac{2}{5}\sqrt{(x+3)^5} - 2\sqrt{(x+3)^3} + C.$$