Expressing graphs as symmetric differences of cliques in the complete graph

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Graphs as symmetric differences of cliques

Let G be a finite simple graph on n vertices.

One can always express G as the *symmetric difference* of a collection of cliques in the complete graph on n vertices.

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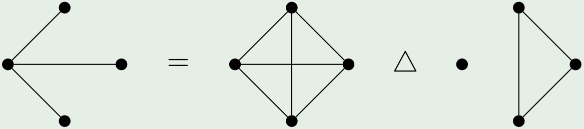
That is, every edge of G appears in an odd number of cliques, every non-edge in an even number.

For instance, take the collection $\{\{u, v\} : uv \in E(G)\}$.
But we can often do better:

Example (The "star strategy")



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Definitions. A *clique construction* of G is a collection $\mathscr{C}$ of subsets of $V(G)$ in which a pair of vertices u, v are adjacent if and only if u and v appear together an odd number of times in $\mathscr{C}$.

The minimum cardinality of a clique construction of G is the *clique-build number* of G, denoted $\mathrm{c}_2(G)$.

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The minimum cardinality of a clique construction of G is the *clique-build number* of G, denoted $\mathrm{c}_2(G)$.

We use "clique-building" terminology not for lack thereof. . . .

Equivalent terminology

The following are equivalent either to taking symmetric differences of cliques or to finding $c_2(G)$.

1. Subgraph complementation.[1]
2. Faithful orthogonal representations (over $\mathbb{F}_2$).[2]
3. Dot product representations (over $\mathbb{F}_2$).[3]
4. Sum modulo 2 of cliques.[4]

[1] M. Kaminski, V. Lozin, and M. Milanic. Recent developments on graphs of bounded clique-width. Discrete Appl. Math. 157(12), 2747–2761 (2009)

[2] L. Lovász. On the Shannon capacity of a graph. IEEE Transactions on Information Theory. 25(1):1–7 (1979)

[3] G. Minton. Dot product representations of graphs. (2008)

[4] V. Vatter, Terminology for expressing a graph as a sum of cliques (mod 2), URL (version: 2018-12-15): https://mathoverflow.net/q/317716

Faithful orthogonal representations

Given a simple graph G and a field $\mathbb{F}$, a *faithful orthogonal representation* of G over $\mathbb{F}$ of dimension d is a map $f : V(G) \to \mathbb{F}^d$ such that

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The minimum dimension of a faithful orthogonal representation of G, denoted $\mathrm{d}(G, \mathbb{F})$, is particularly well-studied over $\mathbb{R}$.

In the case $\mathbb{F} = \mathbb{F}_2$, this problem is equivalent to finding $\mathrm{c}_2(G)$.

What is the equivalence?

Equivalence

Given a clique construction $\mathscr{C} = \{C_1, C_2, \ldots, C_d\}$ of G, assign to each vertex v an incidence vector $\boldsymbol{v} \in \mathbb{F}_2^d$ with entry

$$\boldsymbol{v}_i = \begin{cases} 1 : & v \in C_i; \\ 0 : & \text{otherwise.} \end{cases}$$

Equivalence

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$$\boldsymbol{v}_i = \begin{cases} 1: & v \in C_i; \\ 0: & \text{otherwise.} \end{cases}$$

Two vertices appear together an even number of times in $\mathscr{C}$ if and only if they are represented by orthogonal vectors.

A faithful orthogonal representation of G over $\mathbb{F}_2$ induces a clique construction of G in a similar way.

Upper bounds

By this equivalence, we obtain bounds on $\mathrm{c}_2(G)$ from bounds on $\mathrm{d}(G, \mathbb{F}_2)$, and vice-versa.

For example, it is known that $\mathrm{d}(G, \mathbb{F}_2) \leq n - 2$ when G is not a path.[1]

Thus, for any graph which is not a path,

$$\mathrm{c}_2(G) \leq n - 2.$$

[1] V. Alekseev and V. Lozin. On orthogonal representations of graphs. Discrete Mathematics 226.1-3 (2001): 359-363.

Upper bounds

On the other hand, we can prove new upper bounds on $\mathrm{d}(G, \mathbb{F}_2)$ by bounding $\mathrm{c}_2(G)$.

Theorem (CB, Purcell, Rombach)

For any graph G with vertex cover number $\tau(G)$,

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Idea of proof. Let U be a minimum vertex cover of G. Choose vertices in U one by one, and build the incident edges which have not yet been built in two steps, using the "star strategy" shown in the example.

Incidence matrices

The *clique-incidence matrix*, $M = M(\mathscr{C})$, is the $n \times |\mathscr{C}|$ matrix whose rows are the incidence vectors for the clique construction $\mathscr{C}$.

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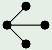
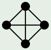
Example

$G =$ , $\mathscr{C} = \Big\{$ ,  $\Big\}$, $M = M(\mathscr{C}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, and

$$MM^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The minimum rank problem

An $n \times n$ matrix A over $\mathbb{F}$ is said to *fit* G if the off-diagonal zeros of A precisely match those of the adjacency matrix of G.

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The *minimum rank of* G *over* $\mathbb{F}$ is the minimum rank over all matrices with entries in $\mathbb{F}$ which fit G, denoted $\text{mr}(G, \mathbb{F})$.

$c_2(G)$ and $\mathrm{mr}(G, \mathbb{F}_2)$

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If $M = M(\mathscr{C})$ is a clique-incidence matrix for G, then MM^T fits G. Since $\mathrm{rank}(MM^T) \leq \mathrm{rank}(M) \leq d$, we obtain the bound

$$\mathrm{mr}(G, \mathbb{F}_2) \leq c_2(G).$$

Additivity

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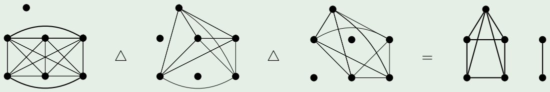
The clique-build number does not behave in the same way.

We can check that $c_2(W_5) = 3$ and $c_2(K_2) = 1$, but we have the following clique construction of $W_5 + K_2$.

Example



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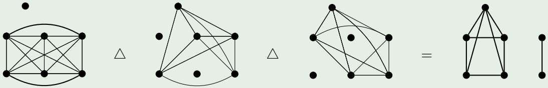

What is special about W_5?

- $c_2(W_5)$ is odd;
- Every vertex of W_5 appears an even number of times in some optimal construction.

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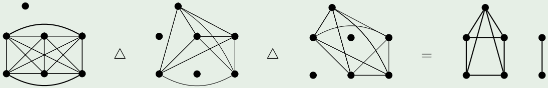

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- $\mathrm{c}_2(W_5)$ is odd;
- Every vertex of W_5 appears an even number of times in some optimal construction.

This tells us that the minimum rank of a graph over $\mathbb{F}_2$ and its clique-build number are not always equal.

In many cases, we do have $\mathrm{c}_2(G) = \mathrm{mr}(G, \mathbb{F}_2)$, *e.g.* forests.

The clique-build number of a forest

The minimum rank of a forest is independent of the field.[1]

[1] N. Chenette, S. Droms, L. Hogben, R. Mikkelson, and O. Pryporova. Minimum rank of a graph over an arbitrary field. The Electronic Journal of Linear Algebra, 2007.

The clique-build number of a forest

The minimum rank of a forest is independent of the field.

Furthermore, the minimum rank problem is solved for forests. It has been reduced to finding the minimum size of a *path cover*, or a collection of disjoint paths which cover the vertex set, $\mathrm{p}(G)$.[1]

Lemma

For any tree T, $\mathrm{mr}(T, \mathbb{R}) = |T| - \mathrm{p}(T)$.

[1] C. Johnson and A. Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. Linear and Multilinear Algebra, 1999.

The clique-build number of a forest

Theorem (CB, Purcell, Rombach)

For any forest G and field $\mathbb{F}$,

$$\mathrm{c}_2(G) = |G| - \mathrm{p}(G) = \mathrm{mr}(G, \mathbb{F}).$$

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Idea of proof.

- $\mathrm{mr}(G, \mathbb{F}_2) = \mathrm{mr}(G, \mathbb{R}) = |T| - p(T) \leq c_2(G)$.
- An algorithm for minimum path covers.[1]
- The star-strategy.

[1] L. Hogben and C. Johnson. Path covers of trees. Pre-print. URL:https://orion.math.iastate.edu/lhogben/research/HJpathcover.pdf.

The clique-build number of a forest

Hogben and Johnson's algorithm for minimum path covers:

1. If T is a spider graph, or generalized star, take a maximal path through the center and all remaining paths.
2. Otherwise, pick off *pendant* spiders one-by-one to obtain an optimal path cover of T.

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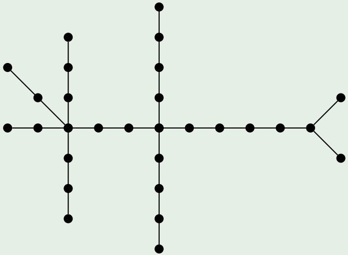

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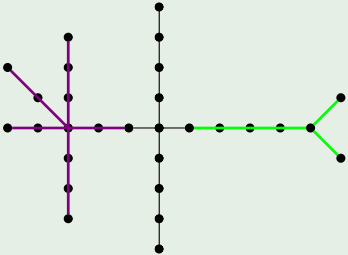

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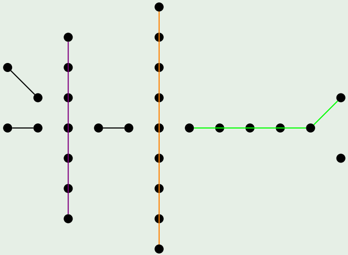

The clique-build number of a forest

Theorem (CB, Purcell, Rombach)

For any forest G and field $\mathbb{F}$,

$$c_2(G) = |G| - p(G) = mr(G, \mathbb{F}).$$

Idea of proof. Let $\mathcal{P}$ be an optimal path cover of G obtained by this algorithm. Note that high degree vertices are internal on their respective paths.

Build the edges of T which lie in $\mathcal{P}$ and which link low-degree vertices one-by-one.

Build the edges incident to each high-degree vertex v in 2 steps using cliques on $N[v]$ and $N(v)$.

This makes a total of $|E(\mathcal{P})| = |G| - p(G)$ cliques, as desired.

$c_2(G)$ and $mr(G, \mathbb{F}_2)$

In general, $c_2(G)$ and $mr(G, \mathbb{F}_2)$ are not always equal, but close.

Theorem (CB, Purcell, Rombach)

For any graph G, either

1. $c_2(G) = mr(G, \mathbb{F}_2)$, *or*
2. $c_2(G) = mr(G, \mathbb{F}_2) + 1$, *in which case* $c_2(G)$ *is odd.*

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Idea of proof.
Let A be a matrix of minimum rank which fits G over $\mathbb{F}_2$.

If $\mathrm{rank}(A)$ is odd, then A decomposes into XX^T for some matrix $X \in \mathbb{F}_2^{n \times k}$, which may be taken as a clique-incidence matrix for G.[1]

[1] S. Friedland and R. Loewy. On the minimum rank of a graph over finite fields. Linear algebra and its applications, 2012.

$c_2(G)$ and $mr(G, \mathbb{F}_2)$

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If $c_2(G) \neq mr(G, \mathbb{F}_2)$, then A does not decompose in this way, so rank(A) is even. Thus, $mr(G + K_2, \mathbb{F}_2) = mr(G, \mathbb{F}_2) + 1$ is odd, so $c_2(G + K_2) = mr(G, \mathbb{F}_2) + 1$ and $c_2(G) \leq mr(G, \mathbb{F}_2) + 1$.

$c_2(G)$ and $\mathrm{mr}(G, \mathbb{F}_2)$

Theorem (CB, Purcell, Rombach)

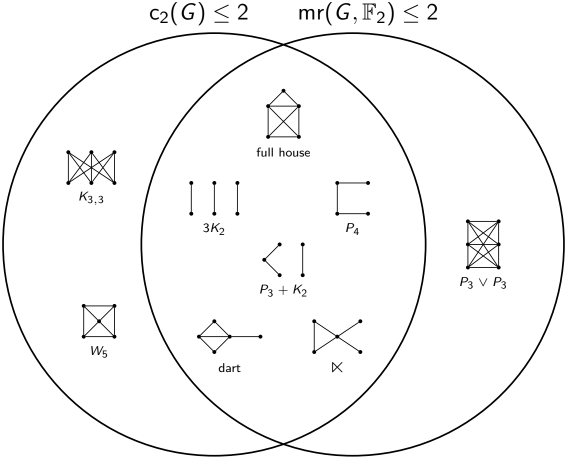
Let G be a graph. The following are equivalent.

1. $c_2(G) = \mathrm{mr}(G, \mathbb{F}_2) + 1$;
2. *there is a unique matrix A of minimum rank over $\mathbb{F}_2$ which fits G, and every diagonal entry of A is 0;*
3. *there is an optimal clique construction of G in which every vertex appears an even number of times;*
4. *for every component G' of G, $c_2(G') = \mathrm{mr}(G', \mathbb{F}_2) + 1$.*

Forbidden induced subgraphs

The property $\mathrm{c}_2(G) \leq k$ is hereditary and finitely defined. For odd k, the sets of minimal forbidden induced subgraphs for $\mathrm{c}_2(G) \leq k$ are the same as those for $\mathrm{mr}(G, \mathbb{F}_2) \leq k$. For even k, this is not so.

Minimal forbidden induced subgraphs



Thank you!