1. This problem concerns root finding.

(a) Use a hand calculator and Newton’s method to find the root of \( f(x) = e^{-x} - \cos x \) closest to \( \pi/2 \). Stop iterating when your first four significant digits are unchanged. How quickly would your solution converge to the true solution of \( f(x) = 0 \) if you continued iterating?

(b) Given the function \( F(x) : \mathbb{R}^n \to \mathbb{R}^n \), set up Newton’s method to find the solution to the equation \( F(x) = 0 \). Be sure to describe how the algorithm will be most efficiently executed (i.e. will you be solving a linear system?). How many floating point operations will be required per iterate?

2. Compute the quadratic polynomial interpolating the data points \((-h, 0), (0, 1), (h, 0)\) using the Newton basis and integrate the result on the interval \([-h, h]\) using the Composite Trapezoid rule at the nodes \( x_0 = -h, x_1 = -h/2, x_2 = 0, x_3 = h/2, x_4 = h \).

3. Given a QR-decomposition of the \( m \times n \) matrix \( A \) where \( m >> n \), i.e. \( A = QR \), describe the steps you would take to compute the solution to the least-squares problem \( Ax = b \). Be sure to explain the role played by the decomposition in the efficiency of the solution as the ratio \( m/n \) gets larger.

4. The general form for the Runge-Kutta method of order 2, to solve the IVP \( y'(t) = f(t, y) \), is given by

\[
Y_{i+1} = Y_i + (ak_1 + bk_2) \\
k_1 = hf(t_i, Y_i) \\
k_2 = hf(t_i + \alpha h, Y_i + \beta k_1)
\]

where \( a, b, \alpha, \beta \) are constants. Derive equations relating these four constants to ensure the method is accurate to order 2. Hint: You may use the Taylor series in two variables:

\[
f(x + h, y + k) = \sum_{i=0}^{\infty} \frac{1}{i!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x, y)
\]

5. Express the height of a projectile thrown from the top of a 200-foot tall building with unknown initial velocity, taking 4 seconds to reach the ground, as a second-order BVP. Solve the BVP using the shooting method with \( h = 1 \) second and forward Euler to solve each IVP.

6. Consider the following unidirectional wave equation:

\[
w_t - g(x) w_x = f(x, t, w), \quad 0 < x < 1, \quad t > 0;
\]
with the initial condition

\[ w(x,0) = w_0(x), \]  

where \( g(x) \), \( f(x,t,w) \), and \( w_0(x) \) are some known functions, with \( g(x) > 0 \) for \( 0 \leq x \leq 1 \). The equation of characteristics for this problem is:

\[ \frac{dx}{dt} = -g(x). \]  

(a) Show that a change of variables

\[ (x,t) \rightarrow (\xi,t), \quad \text{with} \quad \xi = t + \int_x^x \frac{dx'}{g(x')} \]

reduces (1) to

\[ w_t = f(\xi,t,w), \]  

where now the partial derivative w.r.t. \( t \) is taken when \( \xi = \text{const} \).

(b) State where the boundary conditions can be specified. Then set up one time step of the numerical solution of this problem by the Method of Characteristics. Describe all relevant details of the implementation.

7. Consider the Heat equation \( u_t = u_{xx} \) on the domain \( 0 \leq x \leq 1 \) and subject to the Dirichlet boundary conditions. For future reference, a first-order in time, explicit scheme for this equation is:

\[ U_{m+1}^n - U_m^n = r \left( U_{m+1}^n - 2U_m^n + U_{m-1}^n \right), \]  

where \( r = \kappa/h^2 \) and \( \kappa \) and \( h \) are the step sizes in time and space, respectively. The amplification coefficient, obtained by the von Neumann analysis, of one time step of scheme (1) equals:

\[ \rho = 1 - 4r \sin^2 \frac{\beta h}{2}, \]  

where \( \beta \) is the wavenumber of the Fourier harmonic \( \exp[i\beta mh] \). The stability condition for scheme (1) is: \( r \leq 0.5 \).

(a) Write out a fully implicit, first-order in time scheme for the Heat equation. (Just stating the scheme is enough; you do not need to prove that it is first-order accurate.) Use the von Neumann analysis to derive its amplification coefficient of one time step and determine the stability condition for this scheme.

(b) Suppose your initial condition is a discontinuous function of \( x \). Which scheme, (1) or the one you have derived in part (a), will more efficiently smoothen out the discontinuity? Answer this question for each of the two values of \( r \): \( r = 0.2 \) and \( r = 0.4 \).