REAL AND COMPLEX ANALYSIS PHD QUALIFYING EXAM

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The test has two sections, covering real and complex analysis. In order to pass, you must do at least 2 problems from each section **completely correctly**, and you must do a total of 6 problems completely correctly, or 5 completely correctly with substantial progress on 2 others. Some problems have more than one part (e.g., problem 1 in Section I consists of 1a), 1b), and 1c)).

I. REAL ANALYSIS.

1a) State what it means for a function $f : [a, b] \mapsto \mathbf{R}$ to be Riemann integrable on $[a, b]$.
1b) State what it means for a sequence of functions $f_n : [a, b] \mapsto \mathbf{R}$ to converge uniformly to a function $f : [a, b] \mapsto \mathbf{R}$.
1c) Let $f_n : [a, b] \mapsto \mathbf{R}$ be a sequence of functions converging uniformly to some $f : [a, b] \mapsto \mathbf{R}$, and suppose that each f_n is Riemann integrable on $[a, b]$. Show that f is Riemann integrable on $[a, b]$ and that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Solution (part c) only).
Given $\epsilon > 0$, let n be such that $|f(x) - f_n(x)| < \epsilon$ for all $x \in [a, b]$. Then, for any partition P, $U(P, f) \le U(P, f_n) + \epsilon(b - a)$ and $L(P, f) > L(P, f_n) - \epsilon(b - a)$. Let P be a partition such that

$$\int_a^b f_n(x)\,dx - \epsilon < L(P, f_n) \le U(P, f_n) < \int_a^b f_n(x)\,dx + \epsilon.$$

Then $L(P, f) > \int_a^b f_n(x)\,dx - \epsilon(1 + (b - a))$ and $U(P, f) < \int_a^b f_n(x)\,dx + \epsilon(1 + (b - a))$, implying $U(P, f) - L(P, f) < 2\epsilon(1 + (b - a))$, which is Riemann's Criterion for integrability. For the limit result, we note that

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \sup_{[a,b]} |f(x) - f_n(x)|(b - a),$$

which goes to 0 as $n \to \infty$.

2. Suppose that $f : \mathbf{R} \mapsto \mathbf{R}$ satisfies $|f(x) - f(y)| \le |x - y|^{5/4}$ for all x and y in $\mathbf{R}$. Show that f is constant.
Solution.
For any $x \in \mathbf{R}$,

$$\begin{aligned} |f'(x)| &\le \limsup_{h\to 0} \frac{|f(x) - f(x + h)|}{|h|} \\ &\le \lim_{h\to 0} |h|^{1/4} = 0, \end{aligned}$$

which implies that f is constant.

3. Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions defined on $\mathbf{R}$, and that, for all n,

$$\int_{\mathbf{R}} |f_n(x)|^2 \, dm(x) \leq 1,$$

where we are using m to denote Lebesgue measure. Show that, for all $\epsilon > 0$, there is a $\delta > 0$ such that, if $E \subset \mathbf{R}$ is any Lebesgue measurable set satisfying $m(E) < \delta$, then

$$\int_E |f_n(x)| \, dm(x) < \epsilon.$$

Solution. Let χ_E be the characteristic function of the set E. If $m(E) < \infty$, then

$$\begin{aligned} \int_E |f_n(x)| \, dm(x) &= \int |f_n(x)| \, \chi_E(x) \, dm(x) \\ &\leq \|f_n\|_2 \|\chi_E\|_2 \\ &\leq \sqrt{m(E)}. \end{aligned}$$

4a) Find, with justification, the radius of convergence of the power series:

$$\sum_0^\infty 2^{-\sqrt{n}} x^n.$$

4b) Find, with justification, the radius of convergence of the power series:

$$\sum_0^\infty \frac{x^n}{1 + 2 + 4 + \cdots + 2^n}.$$

You may use without proof the standard tests (ratio, root, etc.) for computing radius of convergence, as well as the values of limits taught in elementary calculus: e.g., that $n^{-1} \log n \to 0$ as $n \to \infty$.

Solutions.
4a) Root Test: $(2^{-\sqrt{n}})^{1/n} = 2^{-1/\sqrt{n}} \to 1$ as $n \to \infty$. The radius is 1.
4b) Since $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$, the Root or Ratio Test shows that the radius is 2.

5. Consider the two functions $f_1(x, y, z) \equiv x^2 - 2x + y^2$ and $f_2(x, y, z) \equiv x^2 + y^2 + z^2 - 4$, each mapping from $\mathbf{R}^3$ into $\mathbf{R}$. (Note that f_1 does *not* depend on z.) Define

$$\Sigma \equiv \{(x, y, z) \in \mathbf{R}^3 : \; f_1(x, y, z) = f_2(x, y, z) = 0\}.$$

At what points (x', y', z') on Σ does the Implicit Function Theorem *not* guarantee the existence of an open neighborhood U of z' and differentiable functions $g : U \mapsto \mathbf{R}$ and $h : U \mapsto \mathbf{R}$ such that $(g(z'), h(z'), z') \in \Sigma$ for all $z' \in U$? You do not need to sketch Σ, but it will probably help you to do so.

Solution. With z fixed, the Jacobian matrix of (f_1, f_2) with respect to x and y is

$$\begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 2x-2 & 2y \\ 2x & 2y \end{bmatrix},$$

which is singular if and only if $y = 0$. Setting $y = 0$ and solving for x and z, we see that the 3 "bad" points are $(0,0,2)$, $(0,0,-2)$, and $(2,0,0)$.

6. Find, using the appropriate limit theorem or theorems,

$$\lim_{n\to\infty} \int_0^\infty \frac{x^{1/n}}{(e^x + \frac{x}{n})}\, dm(x).$$

Solution. For all $x > 0$ and all n,

$$\left| \frac{x^{1/n}}{(e^x + \frac{x}{n})} \right| \le (x+1)e^{-x},$$

which is integrable; and

$$\lim_{n\to\infty} \frac{x^{1/n}}{(e^x + \frac{x}{n})} = e^{-x}$$

for all $x > 0$. By Dominated Convergence, the value of the limit is $\int_0^\infty e^{-x}\, dm(x) = 1$.

II. COMPLEX ANALYSIS.

In this section, D always denotes the set $\{z \in \mathbf{C} : |z| < 1\}$.

1. Use residues to show that, for all $a \ge 0$,

$$\int_{-\infty}^\infty \frac{\cos(ax)}{1+x^2}\, dx = \pi e^{-a}.$$

Solution. Set $f(z) = e^{iaz}/(1+z^2)$. For any $R > 0$,

$$\int_{-R}^R f(x)\, dx = \int_{-R}^R \frac{\cos(ax)}{1+x^2}\, dx,$$

because the (odd) term with $i\sin(ax)$ cancels. For $R >> 1$ let γ_R be the contour that runs from $-R$ to R, and then along the circular arc $z = Re^{it}$ for $0 \le t \le \pi$. The function f has a simple pole at $z = i$, where it has a residue equal to

$$\frac{e^{i^2 a}}{(z+i)} = \frac{e^{-a}}{2i}.$$

Therefore

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a}.$$

As $R \to \infty$, the integral from $-R$ to R converges to the integral we want to compute. It's enough to show that the integral of f over the curved part of the contour goes to 0 as $R \to \infty$. On this curve, $|f(z)| \le 1/(R^2-1)$, since $|e^{iaz}| \le 1$ there. Therefore

$$\left| \int_{(\text{curved part})_R} f(z)\,dz \right| \le \pi R/(R^2-1) \to 0$$

as $R \to \infty$, and the result follows.

2. How many zeroes, counting multiplicities, does $f(z) \equiv 3z^2 + e^z$ have inside D?

Solution. The function f is analytic on a neighborhood of $\bar{D}$. Since $|3z^2| \equiv 1$ and $|e^z| \le e < 3$ on all of ∂D, Rouché implies that $3z^2 + e^z$ and $3z^2$ have the same number of zeroes (counting multiplicities) in D. But $3z^2$ has 1 zero of multiplicity 2. Therefore $f(z)$ must have 2 zeroes (counting multiplicities).

3a) State and sketch a proof of Morera's Theorem.
3b) Let $f : D \mapsto \mathbf{C}$ be continuous on all of D, and suppose that f is analytic on $D \setminus \{0\}$. Use Morera's Theorem and Cauchy's Theorem (don't prove Cauchy's Theorem) to show that f is analytic on all of D.

Solution. 3a) *Statement:* Suppose f is continuous on an open disk Δ and $\int_T f(z)\,dz = 0$ for all triangles $T \subset \Delta$. Then f is analytic on Δ. *Sketch of proof:* WLOG, $\Delta = D$. For $z \in D$, set

$$F(z) \equiv \int_{[0,z]} f(\zeta)\,d\zeta,$$

where $[0,\zeta]$ is the directed line segment running from 0 to ζ. If z_1 and z_2 belong to D, then

$$F(z_1) - F(z_2) + \int_{[z_1,z_2]} f(\zeta)\,d\zeta = 0,$$

by our hypothesis on f. Therefore, if $z_1 \ne z_2$,

$$\frac{F(z_2)-F(z_1)}{z_2-z_1} = \frac{1}{z_2-z_1}\int_{[z_1,z_2]} f(\zeta)\,d\zeta.$$

We can parameterize the path $[z_1,z_2]$ as $\gamma(t) = z_1 + (z_2-z_1)t$ for $0 \le t \le 1$. When we do so, we see that

$$\frac{1}{z_2-z_1}\int_{[z_1,z_2]} f(\zeta)\,d\zeta = \int_0^1 f(z_1 + (z_2-z_1)t)\,dt,$$

which approaches $f(z_1)$ as $z_2 \to z_1$, because f is continuous. Therefore $f(z) = F'(z)$ on all of D, implying that F—and thus f—is analytic. 3b) Let T be any triangle in D. If 0 isn't in or on T then $\int_T f(z)\,dz = 0$ by Cauchy's Theorem. If 0 lies on T then $\int_T f(z)\,dz$ equals a limit of contour integrals that "go around" 0, and are all equal to 0, by Cauchy's Theorem; therefore, $\int_T f(z)\,dz = 0$. If 0 is inside T, $\int_T f(z)\,dz$ equals the sum of the (correctly oriented) integrals over the 3 triangles formed by connecting 0 with T's corners, which are all 0 by the previous case. By Morera's Theorem, f is analytic.

4. Consider the function $u(x,y) \equiv e^x \cos y + x^3 - 3xy^2$, which maps from $\mathbf{R}^2$ into $\mathbf{R}$. Show that u is harmonic on $\mathbf{R}^2$, and find a harmonic $v : \mathbf{R}^2 \mapsto \mathbf{R}$ such that

$$f(z) \equiv f(x+iy) = u(x,y) + iv(x,y)$$

is analytic on all of $\mathbf{C}$.

Solution. Showing $u_{xx} + u_{yy} = 0$ is a quick computation. The harmonic conjugate v is $e^x \sin y + 3x^2 y - y^3$.

5. Let

$$f(z) = \frac{z}{z^2 - z - 2}.$$

This function has a Laurent series expansion of the form

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n, \tag{1}$$

valid for all z in the annulus $\{z \in \mathbf{C} :\ 1 < |z| < 2\}$. Compute the coefficients c_n.

Solution. By partial fractions,

$$\frac{z}{z^2 - z - 2} = \frac{1}{3}\left(\frac{1}{z+1}\right) + \frac{2}{3}\left(\frac{1}{z-2}\right).$$

In the annulus,

$$\frac{1}{z+1} = \frac{1}{z}\left(\frac{1}{1+1/z}\right) = \frac{1}{z}\sum_{0}^{\infty}(-1)^k z^{-k}$$

and

$$\frac{1}{z-2} = -\frac{1}{2}\left(\frac{1}{1-z/2}\right) = -\frac{1}{2}\sum_{0}^{\infty}(z/2)^k.$$

Therefore,

$$c_n = \begin{cases} (1/3)(-1)^{n+1} & \text{if } n < 0; \\ -(1/3)2^{-n} & \text{if } n \geq 0. \end{cases}$$

6. Let $f : \mathbf{C} \mapsto \mathbf{C}$ be entire, and suppose that $|f(z)| \leq 13(1+|z|)^{5.3}$ for all $z \in \mathbf{C}$. Show that f is a polynomial of degree no larger than 5.

Solution. Write

$$f(z) = \sum_{0}^{\infty} c_n z^n,$$

where the power series converges for all z, because f is entire. Let M_R equal the maximum value of $|f|$ for $|z| = R$. By hypothesis, $M_R \leq 13(1+R)^{5.3}$. By the Cauchy estimates,

$$|c_n| \leq M_R R^{-n} \leq 13(1+R)^{5.3} R^{-n},$$

which goes to 0 as $R \to \infty$ for all $n \geq 6$. Therefore $c_n = 0$ for all $n \geq 6$, and f is a polynomial of degree ≤ 5.