Unstable-cavity sensitivity to spatially localized intracavity phase aberrations

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The geometric theory of the aberration sensitivity of an unstable cavity to a spatially localized intracavity phase distortion is provided. Such a localized perturbation source could be caused by a small imperfection in some intracavity mirror (such as a deformable mirror) or by a turbulence zone in the flow field of a gas-laser gain medium. This geometric theory, first developed by Anan’ev [Sov. J. Quantum Electron. 1, 565 (1972)] for a uniformly extended intracavity aberration source completely filling the cavity, yields the cavity phase-weighting coefficients that determine the resultant phase-aberration structure outcoupled from the unstable resonator. The analysis presented here provides the dependence of these coefficients on the cavity magnification, the axial position (with respect to the feedback mirror), the transverse location (with respect to the unperturbed optic axis) of the aberration source, the transverse dimension of the source, and the aberration structure characterizing the source.

1. GEOMETRY OF THE ABERRATION STRUCTURE

For convenience, attention will be restricted to a single transverse dimension of a positive-branch confocal unstable cavity whose usual geometric aberration-sensitivity coefficients are given by

\[ \bar{a}_k^G(z) = \frac{1}{M^k - 1} \left\{ M^k + \left[ 1 + (M - 1) \frac{z}{z_T} \right] \right\} \]  

(1.1)

when the aberration source, situated a distance \( z \) from the convex-mirror-feedback aperture plane of the cavity, spans the entire transverse dimension of the cavity.

Let the spatially localized aberration source be situated along a transverse line located the distance \( z \) from the convex-mirror-feedback aperture plane of the cavity centered at \( x_c = \epsilon_c M \alpha_1 \) along this line, with transverse extent \( 2\Delta x = 2\Delta \epsilon \epsilon_M \alpha_1 \), as shown in Fig. 1, so that the aberration source is situated within the closed interval \( [x_c - \Delta x, x_c + \Delta x] = [\alpha_1 (\epsilon_c - \Delta \epsilon), \alpha_1 (\epsilon_c + \Delta \epsilon)] \). The phase-aberration function \( \phi(x, z) \) associated with this spatially localized aberration source may then be expanded in a Taylor series in the transverse coordinate about the centroid position \( x_c \), so that

\[ \phi(x, z) = \sum_{k=0}^{\infty} \delta_k(z_c) (x - x_c)^k; \quad x_c - \Delta x \leq x \leq x_c + \Delta x \]  

(1.2)

and is zero elsewhere, where each aberration coefficient \( \delta_k \) is measured from the center to the edge of the aberration domain. Notice that this aberration expansion includes the zeroth-order aberration \( (k = 0) \), whereas that order is not included in the usual situation in which the aberration source spans the entire transverse extent of the cavity mode.

In the usual situation the zero-order aberration merely retards the phase of the entire mode uniformly and consequently does not result in an aberrated phase structure for that single cavity field; indeed, as is readily evident from Eq. (1.1), the geometric aberration sensitivity to a zeroth-order aberration source that spans the entire field is indeterminate. However, in the present situation the zeroth-order aberration component retards only a portion of the incident field and hence does result in an aberrated cavity field distribution.

In the geometric approximation the deformation of the outcoupled cavity mode phase front resulting from the cumulative effect of the intracavity phase-aberration structure is given by the sum of the optical-path-length differences along the appropriate mode ray trajectories. If the change in optical path length of a ray (associated with the dominant geometric mode of the cavity) during a single round-trip iteration that is due to a given intracavity phase inhomogeneity is denoted by \( \Delta \sigma(x) \), then the optical path difference accrued on the previous round-trip iteration is given by \( \Delta \sigma(x/M) \). The total optical-path difference accumulated by
Phase Aberration Source Plane

\[ \Delta l_k(x) = z \phi_k(x) \left( x - x_c \right)^k; \]
\[ x_c - \Delta x \leq x \leq x_c + \Delta x, \quad x' < x_c - \Delta x; \quad (1.6c) \]

and if the aberration source is intersected only in the expanding pass of that iteration, then

\[ \Delta l_k(x) = z \phi_k(x) \left( \frac{x}{M} \right)^{k-1} \]
\[ x_c - \Delta x \leq x' \leq x_c + \Delta x, \quad x > x_c + \Delta x, \quad (1.6d) \]

where \( x' \) is given in Eq. (1.4). Application of the binomial theorem to each of these expressions results in the following set of equations:

\[ \Delta l_k(x) = 0; \quad x, x' > x_c + \Delta x \quad \text{or} \quad x, x' < x_c - \Delta x, \quad (1.7a) \]
\[ \Delta l_k(x) = z \phi_k(x) \sum_{l=0}^{h} (-1)^l \binom{h}{l} x^{h-l}; \]
\[ x_c - \Delta x \leq x, x' \leq x_c + \Delta x, \quad (1.7b) \]
\[ \Delta l_k(x) = z \phi_k(x) \sum_{l=0}^{h} (-1)^l \binom{h}{l} \frac{1}{M^{h-l}} \]
\[ \times \left[ 1 + z \left( \frac{1}{M} \right)^{h-l} \right] x^{h-l}; \]
\[ x_c - \Delta x \leq x', x' < x_c + \Delta x, \quad (1.7c) \]
\[ \Delta l_k(x) = z \phi_k(x) \sum_{l=0}^{h} (-1)^l \binom{h}{l} \frac{1}{M^{h-l}} \]
\[ \times \left[ 1 + z \left( \frac{1}{M} \right)^{h-l} \right] x^{h-l}; \]
\[ x_c - \Delta x \leq x, x' < x_c + \Delta x, \quad x > x_c + \Delta x, \quad (1.7d) \]

where the \( \binom{j}{k} \) are the binomial coefficients. The total optical path difference is then given by

\[ \Delta l_{\text{TOT}}(x) = \sum_{p=0}^{m} \sum_{k=0}^{h} \Delta l_k \left( \frac{x}{M^p} \right); \]
\[ \text{where the summation over } p \text{ must be broken up into the various regions described in Eqs. (1.6) or (1.7).} \]

Let \( P \) denote the number of round-trip iterations it takes for a ray to propagate back into the cavity from the outcoupling aperture such that on the next iteration it will first intersect the localized aberration source (i.e., either its \( x \) or \( x' \) coordinate will be less than \( x_c + \Delta x \) in that next iteration). Its minimum fractional value \( P_{\text{min}} \) is then given by

\[ (x_c + \Delta x)M^{P_{\text{min}}} = a_1 \]

with solution

\[ P_{\text{min}} = \frac{\ln(a_1/(x_c + \Delta x))}{\ln M} = -1 - \frac{\ln(e + \Delta x)}{\ln M}, \quad (1.9) \]

whereas its maximum fractional value is given by

\[ (x_c + \Delta x)M^{P_{\text{max}}} = M a_1 \]

with solution
Let $P'_{\text{max}}$ denote the number of round-trip iterations it takes for a ray to propagate back into the cavity from the outcoupling aperture such that on the next iteration it will cease to intersect the localized aberration source (i.e., either its $x$ or $x'$ coordinate will be less than $x_c - \Delta x$ in that next iteration). In analogy with Eqs. (1.9)–(1.14) one then has

$$P'_{\text{min}} = \frac{\ln[a_i/(x_c - \Delta x)]}{\ln M} = -1 - \frac{\ln(e_c - \Delta e)}{\ln M},$$

$$P'_{\text{max}} = P'_{\text{min}} + 1,$$

$$P' = [P'_{\text{max}}],$$

$$R^{-} = (x_c - \Delta x)M^{P'} = M^{P'+1}a_i(e_c - \Delta e),$$

so that for $Ma_1 \geq x > R^-$ in the outcoupling aperture, it takes $P'$ iterations for a ray to propagate back into the cavity such that on the next iteration its $x$ coordinate is less than $x_c - 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Iteration regions in the exit pupil of the cavity outcoupling aperture for a spatially localized aberration source centered at $x_c$.}
\end{figure}
Fig. 3. Maximum and integer number of iterations necessary for a ray to propagate back into the cavity from the exit pupil of the outcoupling aperture such that on the next iteration it will either first intersect the localized aberration source \(P_{\text{max}}\) or cease to intersect the same aberration source \(P'_{\text{max}}\) as a function of the aberration source centroid parameter \(\epsilon_c = x/M\alpha_1\) for an \(M = 2\) cavity and a relative aberration source width of \(\Delta \varepsilon = 0.05\). 

\[\Delta \varepsilon, \text{ while for } R^+ \geq x \geq \alpha_1, \text{ it takes } P' - 1 \text{ iterations to do so, as illustrated in Fig. 2.}\]

As an example, consider an \(M = 2\) cavity and a localized intracavity aberration source with \(\Delta \varepsilon = 0.05\). The behavior of both the maximum and the integer number of iterations it takes for a ray (associated with the dominant geometric cavity mode) to propagate back into the cavity from the exit pupil of the outcoupling aperture such that on the next iteration it will either first intersect the localized aberration source \(P_{\text{max}}\) or will cease to intersect this same aberration source \(P'_{\text{max}}\) is illustrated in Fig. 3 as a function of the aberration source centroid parameter \(\epsilon_c = x/M\alpha_1\). Note that for sufficiently large values of \(\epsilon_c\), the values of \(P\) and \(P'\) will overlap (viz., \(P = P'\)) over some finite range of values of \(\epsilon_c\) (in this case there are several such regions), while for sufficiently small values of \(\epsilon_c\), the values of \(P\) and \(P'\) will no longer overlap and the inequality \(P > P'\) will hold throughout this region. The corresponding behavior of \(R^+\) and \(R^-\) is depicted in Fig. 4(a) as a function of \(\epsilon_c\), and the resultant behavior of \(\Delta R = R^+ - R^-\) is given in Fig. 4(b). The discontinuous behavior in both \(R^+\) and \(\Delta R\) is merely a consequence of the discontinuous integer behavior of \(P\) or \(P'\). 

Negative values of \(\Delta R\) indicate that the geometric “image” of the aberration source in the exit pupil is folded inside out, as shown in the middle of Fig. 2. Notice that when \(\epsilon_c\) becomes sufficiently small that the inequality \(P > P'\) holds, \(\Delta R\) begins to oscillate rapidly about the value zero. In this region, \(P\) and \(P'\) are sufficiently large (provided that \(\Delta \varepsilon\) is sufficiently small) and may then be approximated by their respective continuous values \(P_{\text{max}}\) and \(P'_{\text{max}}\). In that case Eqs. (1.10), (1.14), (1.16), and (1.19) yield

\[
\frac{\Delta R}{M\alpha_1} = M^P(\epsilon_c + \Delta \varepsilon) - M^{P'}(\epsilon_c - \Delta \varepsilon)
\]

\[
\equiv M^{P'_{\text{max}}}(\epsilon_c + \Delta \varepsilon) - M^{P_{\text{max}}}(\epsilon_c - \Delta \varepsilon).
\]

Since

\[
M^{P_{\text{max}}} = M^{-\ln(\epsilon_c + \Delta \varepsilon)/\ln M} = \frac{1}{\epsilon_c + \Delta \varepsilon},
\]

\[
M^{P'_{\text{max}}} = M^{-\ln(\epsilon_c - \Delta \varepsilon)/\ln M} = \frac{1}{\epsilon_c - \Delta \varepsilon},
\]

so that (rigorously)

\[
M^{P'_{\text{max}}}(\epsilon_c + \Delta \varepsilon) = M^{P'_{\text{max}}}(\epsilon_c - \Delta \varepsilon) = 1,
\]

one obtains the result

\[
\Delta R = R^+ - R^- \approx 0
\]

valid for sufficiently small values of \(\epsilon_c\) and \(\Delta \varepsilon\). Hence, when the aberration source centroid parameter \(\epsilon_c\) becomes sufficiently small that the inequality \(P > P'\) holds and both \(P \approx P_{\text{max}}\) and \(P' \approx P'_{\text{max}}\) for all such values of \(\epsilon_c\) (this guarantees

Fig. 4. Behavior of the localized aberration source image in the exit pupil of the cavity outcoupling aperture as a function of the aberration source centroid parameter \(\epsilon_c\) for an \(M = 2\) cavity and a relative aberration source width of \(\Delta \varepsilon = 0.05\). (a) The relative behavior of the outer \((R^+)\) and inner \((R^-)\) edges of the source image; (b) depicts the resultant difference \(\Delta R = R^+ - R^-\) (the image size).
that $\Delta \epsilon$ is not too large), then the geometric “image” size $\Delta R$ of the aberration source in the cavity exit pupil is a rapidly oscillating function of $\epsilon$, and, on the average, vanishes.

2. GEOMETRIC APPROXIMATION OF THE LOCALIZED ABERRATION SENSITIVITY

Attention is now finally turned to the aberration sensitivity of the unstable cavity to the localized aberration source. In doing so there are two distinct cases to consider. In the first case, illustrated in both the top and bottom diagrams of Fig. 2, the inequality $R^+ > R^-$ is satisfied, and the image of the aberration source in the exit pupil of the cavity has what may be called a normal orientation. In the second case, illustrated in the middle diagram of Fig. 2, the opposite inequality $R^+ < R^-$ is satisfied, and the image of the aberration source in the exit pupil of the cavity has an inverted orientation.

In each case the aberration sensitivity of the cavity is separately defined over three distinct regions of the outcoupling aperture. In the normal orientation these regions are $[a, R^+)$, $[R^-, R^+]$, and $(R^+, M_A)$, whereas in the inverted orientation they are $[a, R^+]$, $(R^+, R^-)$, and $(R^-, M_A)$. In each of these regions the aberration sensitivity is determined in part by both the associated number of iterations $P$ it takes before the aberration source is first intersected and the number of iterations $P$ it takes until the aberration source is no longer intersected.

It will be assumed here that each of these numbers of iteration counts will be constant over a given characteristic region of the outcoupling aperture. This will be strictly true only if the aberration source is located at the feedback aperture plane of the cavity. In general, it is difficult to determine the error incurred in imposing this assumption (just as it is overly cumbersome to specify rigorously the aberration sensitivity within each region of a further subdivision of the outcoupling aperture, which would account for additional single interactions with the aberration source dependent on the aberration source location and size); however, this error should be entirely negligible if the aberration source dimension were small in comparison with the transverse cavity dimension. Even for larger aberration source dimensions, the results obtained here should remain a reasonably good approximation to the true geometric behavior. In either case the aberration sensitivity obtained here will represent some average of the true aberration-sensitivity behavior in each of the three subregions of the cavity outcoupling aperture.

Over the entire exit pupil of the outcoupling aperture the outcoupled phase-aberration structure may be written as

$$\Phi_{\text{OUT}}(x) = \sum_{m=0}^{\infty} \delta_m^\text{OUT} x^m, \quad (2.1)$$

where $\delta_m^\text{OUT}$ is the outcoupled aberration strength of order $m$ measured relative to the (unperturbed) optic axis of the cavity. From the preceding discussion and the structure of Eq. (1.7), it is expected that each order outcoupled aberration strength $\delta_m^\text{OUT}$ is linearly related to the applied intracavity aberration strength $\delta_k(z_1)$ by an expression of the form

$$\delta_m^\text{OUT} = \sum_{k=m}^{\infty} (M_{k})^{k-m} \delta_k(z_1) \alpha_{m,k}(z, \epsilon, \Delta \epsilon). \quad (2.2)$$

Notice that each outcoupled aberration strength of order $m$ is given by a weighted sum over all applied aberration strengths of order $m$ and greater present in the localized intracavity aberration source. This expansion reduces to the usual form described in Ref. 2 when the aberration source centroid $x_1$ is on the optic axis (viz, when $x_1 = 0$), as is readily seen from Eq. (1.7). Hence Eq. (2.2) is simply a generalization of the usual form of the outcoupled aberration strengths necessary to account for the off-axis nature of the aberration source centroid location. The (generalized) coefficients $\alpha_{m,k}$ are simply the localized aberration-sensitivity coefficients for the cavity. The factor $(M_k)^{k-m}$ appearing here is necessary to preserve the proper dimensionality of the expression with $\alpha_{k,m}$ being dimensionless (by definition). The geometric construction of these localized aberration-sensitivity coefficients in each of the characteristic subregions of the cavity outcoupling aperture is now considered.

A. Case 1: Normal Orientation ($R^+ > R^-$)

1. Outer Region ($R^+ < x \leq M_A$)

Within the outer region of the outcoupling aperture a ray will first intersect the aberration source on a converging pass through the cavity after $P$ iterations (see Fig. 2). The change in optical path length due to this first interaction is then given by Eq. (1.7d). The ray will last intersect the aberration source on a collimated pass through the cavity on the $P$th iteration. The change in optical path length due to this last interaction is then given by Eq. (1.7c). Between the first and last interactions the ray intersects the aberration source on both the collimated and the converging passes through the cavity, and the associated change in optical path length for a ray in the outer region of the outcoupling aperture ($R^+ < x \leq M_A$) is then given by

$$\Delta \epsilon_{\text{OUT}}(x) = z_T \left( \sum_{p=P+2}^{P-1} \sum_{h=0}^{k} \left[ \left( \sum_{i=0}^{l} (M_{k})^{l-i} \right) \right] \right) \frac{1}{M^{l-i}} \frac{1}{1 + \left( \frac{z}{z_T} \right)^{l-i}}$$

This generalization of the usual form described in Ref. 2 when the aberration source centroid $x_1$ is on the optic axis (viz, when $x_1 = 0$), as is readily seen from Eq. (1.7). Hence Eq. (2.2) is simply a generalization of the usual form of the outcoupled aberration strengths necessary to account for the off-axis nature of the aberration source centroid location. The (generalized) coefficients $\alpha_{m,k}$ are simply the localized aberration-sensitivity coefficients for the cavity. The factor $(M_k)^{k-m}$ appearing here is necessary to preserve the proper dimensionality of the expression with $\alpha_{k,m}$ being dimensionless (by definition). The geometric construction of these localized aberration-sensitivity coefficients in each of the characteristic subregions of the cavity outcoupling aperture is now considered.
The summation over \( p \) is
\[
\sum_{p=P+2}^{P-1} \frac{1}{M^{k-1p}} = \sum_{p=0}^{P} \frac{1}{M^{k-1p}} - \sum_{p=0}^{P+1} \frac{1}{M^{k-1p}} = \frac{1}{(M^{k-1}-1)M^{k-1(P+1)}} \left[ 1 - \frac{1}{M^{k-1}(P-P-2)} \right].
\]
Substitution of this result into the above equation then yields
\[
\Delta_{\text{TOT}}(x) = z_T \sum_{k=0}^{\infty} \delta_k(x) \sum_{i=0}^{k} \frac{1}{M^{k-1}(P+1)} \left[ 1 - \frac{1}{M^{k-1}(P-P-1)} \right] \tilde{\alpha}_{k-l}(z)
\]
\[
- \frac{1}{M^{k-1}(P+1)} + \frac{1}{M^{k-1}P},
\]
(2.3)
where \( \tilde{\alpha}_{k-l}(z) \) denotes the usual geometric phase-weighting coefficient (or aberration sensitivity) of the cavity for an aberration source that spans the entire transverse dimension of the cavity, given in Eq. (1.1). Notice that if a single-order \( k \) aberration source is introduced into the cavity with a nonvanishing centroid position \( x_0 \), the outcoupled phase-aberration structure (which is measured with respect to the unperturbed optic axis of the cavity) will be composed of all aberration orders up through and including \( k \). That is, each outcoupled aberration order \( m \) is due to all aberrations of order \( m \) and greater present in the localized aberration source. Comparison of the above expression with the expansion given in Eq. (2.2) shows that the localized aberration-sensitivity coefficients for the cavity are given by
\[
\tilde{\alpha}_{m,k}(z, \epsilon, \Delta \epsilon) = (-1)^{k-m} \frac{k!}{m!(k-m)!} \epsilon_k^{k-m}
\]
\[
\times \frac{1}{M^{m(P+1)}} \left[ 1 - \frac{1}{M^{m(P-P-1)}} \tilde{\alpha}_{k-l}(z) \right] - \frac{1}{M^{m(P+1)} + \frac{1}{M^{mP}}},
\]
(2.4)
for \( R^+ < x \leq Ma_1 \) in the normal orientation.

2. Central Region \((R^- \leq x \leq R^+)\)
Within the central region of the outcoupling aperture a ray will first intersect the aberration source on a converging pass through the cavity after \( P - 1 \) iterations, and the ray will last intersect the aberration source on a collimated pass through the cavity on the \( P' \)th iteration. The localized aberration-sensitivity coefficients for the cavity in this region may then be obtained from Eq. (2.7) by replacing \( P' \) by \( P - 1 \), yielding
\[
\tilde{\alpha}_{m,k}(x, \epsilon, \Delta \epsilon) = (-1)^{k-m} \frac{k!}{m!(k-m)!} \epsilon_k^{k-m}
\]
\[
\times \frac{1}{M^{m(P+1)}} \left[ 1 - \frac{1}{M^{m(P-P-1)}} \tilde{\alpha}_m(z) \right] - \frac{1}{M^{m(P+1)} + \frac{1}{M^{mP}}},
\]
(2.5)
for \( R^- \leq x \leq R^+ \) in the normal orientation.

3. Inner Region \((a_1 \leq x < R^-)\)
Within the inner region of the outcoupling aperture a ray will first intersect the aberration source on a converging pass through the cavity after \( P - 1 \) iterations, and the ray will last intersect the aberration source on a collimated pass through the cavity on the \( (P - 1) \)st iteration. The localized aberration-sensitivity coefficients for the cavity in this region may then be obtained from Eq. (2.5) by replacing \( P' \) by \( P - 1 \), yielding
\[
\tilde{\alpha}_{m,k}(x, \epsilon, \Delta \epsilon) = (-1)^{k-m} \frac{k!}{m!(k-m)!} \epsilon_k^{k-m}
\]
\[
\times \frac{1}{M^{m(P+1)}} \left[ 1 - \frac{1}{M^{m(P-P-1)}} \tilde{\alpha}_m(z) \right] - \frac{1}{M^{m(P+1)} + \frac{1}{M^{mP}}},
\]
(2.6)
for \( a_1 \leq x < R^- \) in the normal orientation.

B. Case 2: Inverted Orientation \((R^- > R^+)\)
1. Outer Region \((R^- \leq x \leq Ma_1)\)
Within the outer region of the outcoupling aperture a ray will first intersect the aberration source on a converging pass through the cavity after \( P \) iterations and will last intersect the aberration source on a collimated pass through the cavity on the \( P \)th iteration. The localized aberration-sensitivity coefficients for the cavity in this region are then given by Eq. (2.4), viz.,
\[
\tilde{\alpha}_{m,k}(z, \epsilon, \Delta \epsilon) = (-1)^{k-m} \frac{k!}{m!(k-m)!} \epsilon_k^{k-m}
\]
\[
\times \frac{1}{M^{m(P+1)}} \left[ 1 - \frac{1}{M^{m(P-P-1)}} \tilde{\alpha}_m(z) \right] - \frac{1}{M^{m(P+1)} + \frac{1}{M^{mP}}},
\]
(2.7)
for \( R^- \leq x \leq Ma_1 \) in the inverted orientation.

2. Central Region \((R^+ < x < R^-)\)
Within the central region of the outcoupling aperture a ray will first intersect the aberration source on a converging pass through the cavity after \( P \) iterations and will last intersect the aberration source on a collimated pass through the cavity on the \( (P - 1) \)st iteration. The localized aberration-sensitivity coefficients for the cavity in this region may then be obtained from Eq. (2.7) by replacing \( P' \) by \( P - 1 \), yielding
\[
\tilde{\alpha}_{m,k}(x, \epsilon, \Delta \epsilon) = (-1)^{k-m} \frac{k!}{m!(k-m)!} \epsilon_k^{k-m}
\]
\[
\times \frac{1}{M^{m(P+1)}} \left[ 1 - \frac{1}{M^{m(P-P-1)}} \tilde{\alpha}_m(z) \right] - \frac{1}{M^{m(P+1)} + \frac{1}{M^{mP}}},
\]
(2.8)
for \( R^+ < x < R^- \) in the inverted orientation.

3. Inner Region \((a_1 \leq x \leq R^+)\)
Within the inner region of the outcoupling aperture a ray will first intersect the aberration source on a converging pass
through the cavity after $P - 1$ iterations and will last intersect the aberration source on a collimated pass through the cavity on the $(P' - 1)$st iteration. The localized aberration-sensitivity coefficients for the cavity in this region may then be obtained from Eq. (2.8) by replacing $P$ by $P - 1$, yielding

$$
\tilde{\alpha}_{m,k}(z, \epsilon, \Delta\epsilon) = (-1)^{k-m} \frac{k!}{m!(k-m)!} \epsilon^{k-m} \times \left\{ \frac{1}{M_{mp}} \left[ 1 - \frac{1}{M_{mp}(P'-1)} \right] \tilde{\alpha}_{m}^G(z) - \frac{1}{M_{mp}^2} + \frac{1}{M_{mp}(P'-1)} \right\},
$$

(2.9)

for $a_1 \leq x \leq R^+$ in the inverted orientation.

3. DISCUSSION

With these results the outcoupled phase-aberration structure due to a localized intracavity aberration source may be directly obtained over the entire exit pupil of the cavity. In each of the defined geometric subregions of the outcoupling aperture (which are characteristic of the transverse position and shape of the intracavity aberration source) the outcoupled $m$th-order aberration strength is given by Eq. (2.2), which may be written in matrix notation as

$$
\delta^\text{OUT} = \mathbb{A}\delta.
$$

(3.1)

Here $\delta$ and $\delta^\text{OUT}$ are each $(\infty \times 1)$ column vectors with elements $\delta_m$ and $\delta_m^\text{OUT}$, respectively, where

$$
\delta = \begin{pmatrix}
\delta_0 \\
\delta_1 \\
\delta_2 \\
\vdots
\end{pmatrix}, \quad \delta^\text{OUT} = \begin{pmatrix}
\delta_0^\text{OUT} \\
\delta_1^\text{OUT} \\
\delta_2^\text{OUT} \\
\vdots
\end{pmatrix}
$$

(3.2)

and where $\mathbb{A}$ is the $(\infty \times \infty)$ aberration-sensitivity matrix of the cavity, given by

$$
\mathbb{A} = \begin{pmatrix}
\tilde{\alpha}_{0,0} & (M_{a1})\tilde{\alpha}_{0,1} & (M_{a1}^2)\tilde{\alpha}_{0,2} & (M_{a1})\tilde{\alpha}_{0,3} & \cdots \\
0 & \tilde{\alpha}_{1,1} & (M_{a2})\tilde{\alpha}_{1,2} & (M_{a2}^2)\tilde{\alpha}_{1,3} & \cdots \\
0 & 0 & \tilde{\alpha}_{2,2} & (M_{a3})\tilde{\alpha}_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

(3.3)

Notice that for an on-axis aberration source ($\epsilon = 0$), only the diagonal elements of this matrix are nonvanishing.

In general, the determinant of the aberration-sensitivity matrix is nonvanishing and is given by the infinite summation over the on-axis sensitivity coefficients $\tilde{\alpha}_{m,k}$, viz.,

$$
|\mathbb{A}| = \sum_{k=0}^{\infty} \tilde{\alpha}_{k,0}.
$$

(3.4)

However, the convergence of this infinite series is questionable since the first term $\tilde{\alpha}_{0,0}$ is indeterminate over each subregion of the outcoupling aperture [as can easily be seen by inspection of Eqs. (2.4)–(2.9) and noting from Eq. (1.1) that $\tilde{\alpha}_{0}^G = \infty$]. However, as is shown below, this zeroth-order coefficient may be redefined so as to be well behaved over the entire domain of the outcoupling aperture, provided that the aberration source does not completely span the transverse cavity dimension. With this redefinition of the zeroth-order aberration coefficient, each term of the infinite series (3.4) is well behaved. Furthermore, it is readily shown that

$$
\lim_{m \to 0} (\tilde{\alpha}_{k,0})^{1/k} < 1,
$$

(3.5)

provided again that the aberration source does not span the entire transverse cavity domain. As a consequence of Cauchy's convergence test, the infinite series given in Eq. (3.4) is convergent. On the other hand, if the aberration source completely spans the cavity, then the above limit is equal to unity over the entire outcoupling aperture, and the convergence of this series is in question. This difficulty may be circumvented by restricting attention to aberration orders that are less than or equal to some maximum value $k_{\text{max}}$ specified by some other (perhaps system-related) requirement. Hence, in any practical application the inverse of the aberration-sensitivity matrix $\mathbb{A}$ is uniquely specified by Cramer's rule.

Attention is now turned to the behavior of the zeroth-order aberration-sensitivity coefficient $\tilde{\alpha}_{0,0}$. Although this coefficient is indeterminate over each subregion of the outcoupling aperture, the difference between the coefficients in adjacent subregions is shown here to be well behaved. The indeterminacy associated with each of these coefficients is, in a sense, due to the lack of a reference for this particular aberration-sensitivity measure. The procedure of taking the difference between these coefficients in adjacent subregions yields a well-behaved (convergent) solution, since this then provides a reference plane. For convenience, this reference plane will be taken as the central region of the outcoupling aperture (viz., $R^- \leq x \leq R^+$ in the normal orientation and $R^+ < x < R^-$ in the inverted orientation).

Consider first the behavior of the zeroth-order aberration sensitivity in the normal orientation. The difference between the aberration-sensitivity coefficients in the outer and central regions of the outcoupling aperture is, from Eqs. (2.4) and (2.5), given by

$$
\tilde{\alpha}_{m,k}^0 - \tilde{\alpha}_{m,k}^c = (-1)^{k-m} \frac{k!}{m!(k-m)!} \epsilon^{k-m} \times \left[ \frac{1}{M_{mp}} \left( \frac{1}{M^m} - 1 \right) \tilde{\alpha}_{m}^G + \frac{1}{M_{mp}} \left( 1 - \frac{1}{M^m} \right) \right],
$$

(3.6)

which is indeterminate at $m = 0$. However, in the limit as $m$ approaches zero from above, one obtains for $k = 0$

$$
\lim_{m \to 0} (\tilde{\alpha}_{m,0}^0 - \tilde{\alpha}_{m,0}^c) = \lim_{m \to 0} \left[ \frac{1}{M_{mp}(P+1)} \left( 1 - M^m \right) \tilde{\alpha}_{m}^G \right]
$$

$$
= \lim_{m \to 0} \left( \frac{1}{M_{mp}(P+1)} \left( 1 - M^m \right) \frac{1}{M^m - 1} \right)
$$

$$
	imes \left[ M^m + \left( 1 + (M - 1) \frac{x}{z_T} \right) \right] = -2.
$$

Therefore, defining the difference between the zeroth-order aberration-sensitivity coefficients by its limiting behavior, one then has that
Consider next the behavior of the zeroth-order aberration sensitivity in the inverted orientation. The difference between the aberration-sensitivity coefficients in the outer and central regions of the outcoupling aperture is, from Eqs. (2.7) and (2.8), given by

\[
\tilde{\alpha}_{0,0}^0(z, \Delta \epsilon) - \tilde{\alpha}_{0,0}^c(z, \Delta \epsilon) = -2. \tag{3.7}
\]

In the same manner one obtains from Eqs. (2.5) and (2.6) that

\[
\tilde{\alpha}_{0,0}^c(z, \epsilon_c, \Delta \epsilon) - \tilde{\alpha}_{0,0}^c(z, \epsilon_c, \Delta \epsilon) = -2. \tag{3.8}
\]

which is indeterminate at \( m = 0 \). However, in the limit as \( m \) approaches zero from above, one readily obtains for \( k = 0 \)

\[
\lim_{m \to 0} (\tilde{\alpha}_{m,0}^0 - \tilde{\alpha}_{m,0}^c) = 2.
\]

Therefore, defining the difference between the zeroth-order aberration-sensitivity coefficient by its limiting behavior, one has that

\[
\tilde{\alpha}_{0,0}^0(z, \epsilon_c, \Delta \epsilon) - \tilde{\alpha}_{0,0}^c(z, \epsilon_c, \Delta \epsilon) = 2. \tag{3.10}
\]

In the same manner, one obtains from Eqs. (2.8) and (2.9) that

\[
\tilde{\alpha}_{0,0}^c(z, \epsilon_c, \Delta \epsilon) - \tilde{\alpha}_{0,0}^c(z, \epsilon_c, \Delta \epsilon) = 2. \tag{3.11}
\]

Hence, in each case the difference between the aberration-sensitivity coefficients in the three subregions of the outcoupling aperture is equal (in magnitude) to 2. This factor of 2 is simply due to an additional (or subtractional) double interaction with the localized aberration source in a single iteration through the cavity. The behavior of this zeroth-order aberration-sensitivity measure is depicted in Fig. 5 as a function of the aberration source centroid parameter \( \epsilon_c \) with \( \Delta \epsilon = 0.05 \) for an \( M = 2 \) unstable cavity.
NORMAL ORIENTATION

INVERTED ORIENTATION

Fig. 6. Outcoupled aberration structure due to a localized intracavity aberration source with pure defocus.

\[ \Delta x = 0.05 \text{ for an } M = 2 \text{ unstable cavity.} \]

Since the image of the aberration source is situated between \( R^- \) and \( R^+ \) in the normal orientation and is situated between \( a_1 \) and \( R^+ \) and between \( R^- \) and \( M a_1 \) in the inverted orientation in the outcoupling aperture, it is seen that this image always retains the proper phase-shift sense that is characteristic of the aberration source. Nevertheless, in the inverted orientation it appears that the outcoupled phase shift is inverted from that of the aberration source if one mistakenly assumes that the image is centrally located within the outcoupling aperture domain (compare the upper and lower sketches in Fig. 5). Similar remarks hold true in the normal orientation case. Thus, for this simplest of aberrations alone one cannot, in general, uniquely determine the true sense of the aberration source (i.e., whether it is an advancement or a retardation in phase), nor can one uniquely determine its size, from a measurement of the phase structure outcoupled from the cavity. Hence, the inverse problem of determining the intracavity aberration source phase structure from a knowledge of the outcoupled phase structure is nonunique.

Similar remarks hold true for the higher aberration orders. In the normal orientation (\( R^+ > R^- \)), the aberration-sensitivity coefficients satisfy the inequalities

\[
\tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) - \tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) \leq 0, \\
(3.12)
\]

\[
\tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) - \tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) \leq 0, \\
(3.13)
\]

whereas in the inverted orientation (\( R^- > R^+ \)) the aberration-sensitivity coefficients satisfy the opposite inequalities

\[
\tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) - \tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) \geq 0, \\
(3.14)
\]

\[
\tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) - \tilde{a}_{m,k}(z, \epsilon_x, \Delta \epsilon) \geq 0. \\
(3.15)
\]

Because of the structure of the higher-order aberrations \( k \geq 1 \) one can, in principle, determine whether the image of the aberration source is in either the normal or the inverted orientation. From the above two sets of inequalities it is seen that the proper orientation is determined by the region in which the outcoupled aberration strength at any particular nonzero aberration order is the greatest. This behavior is illustrated in Fig. 6 for pure defocus \( (k = 2) \). Hence part of the ambiguity in the inverse problem is removed when higher-order aberrations are present in the intracavity aberration source. However, the ambiguity in the aberration source location and size remain so that the determination of the aberration source strength coefficients from measurements of the outcoupled phase-aberration structure remains nonunique.

REFERENCES