Velocity of energy transport for a time-harmonic field in a multiple-resonance Lorentz medium

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The velocity of propagation of electromagnetic energy for a monochromatic plane-wave field in a causally dispersive dielectric medium with absorption, as described by the Lorentz model, is considered within the framework of Poynting’s theorem. A general, rigorous expression for the energy-transport velocity in a Lorentz medium with multiple-resonance frequencies is derived. From this rigorous result, an approximate expression for the energy velocity is obtained that is in a form that is independent of the medium model and so is likely to be applicable to general dispersive media.

INTRODUCTION

A quantity of fundamental importance to both the analysis and physical interpretation of propagation phenomena in a dispersive medium is the velocity of energy transport (or the energy velocity) of a completely monochromatic plane-wave field. This physical velocity is defined as a ratio of the time-average value of the Poynting vector to the total time-average electromagnetic energy density stored in both the field and the medium. The original derivation of this quantity for a single-resonance Lorentz medium by Brillouin neglected to include that portion of the electromagnetic energy that resides in the excited Lorentz oscillators of the medium and consequently was in error. This was pointed out by Loudon, who was the first to obtain a correct expression for the energy velocity in a single-resonance Lorentz medium. The generalization of this analysis to the more physically realistic case of a multiple-resonance Lorentz medium is now given.

VELOCITY OF ENERGY TRANSPORT FOR A TIME-HARMONIC FIELD IN A MULTIPLE-RESONANCE LORENTZ MEDIUM

The analysis begins with the Lorentz model of linear dispersive media, which describes dielectric-type media as a collection of neutral atoms with elastically bound electrons to the nucleus (i.e., as a collection of Lorentz oscillators). Under the action of an applied electromagnetic field, the equation of motion of a typical bound electron is given by

$$m\left(\frac{d^2\mathbf{r}_j}{dt^2} + 2\omega_j\frac{d\mathbf{r}_j}{dt} + \omega_j^2\mathbf{r}_j\right) = -e\mathbf{E}_{\text{loc}}$$

where \(m\) is the mass of the electron and \(e\) is the magnitude of the electronic charge. The quantity \(\omega_j\) is the undamped resonance frequency of this oscillator type (denoted by the index \(j\)), and \(\delta_j\) is the associated phenomenological damping constant of the oscillator. The applied field \(\mathbf{E}_{\text{loc}}\) is the local microscopic electric field intensity, which acts on the electron as a driving force. Under the action of an applied monochromatic field of angular frequency \(\omega\), we obtain the phasor solution to Eq. (1) as

$$\mathbf{r}_j = \frac{e/m}{\omega^2 - \omega_j^2 + 2i\delta_j\omega} \mathbf{E}_{\text{loc}}$$

and the local induced dipole moment is then given by

$$\mathbf{p}_j = -e\mathbf{r}_j = \frac{-e^2/m}{\omega^2 - \omega_j^2 + 2i\delta_j\omega} \mathbf{E}_{\text{loc}}$$

If there are \(N_j\) Lorentz oscillators per unit volume (\(j = 0, 2, 4, \ldots\)), characterized by resonance frequencies \(\omega_j\) and damping constants \(\delta_j\), then the macroscopic polarization of the medium is given by the summation over all oscillator types as

$$\mathbf{P} = \sum_j N_j \langle \mathbf{p}_j \rangle = \langle \mathbf{E}_{\text{loc}} \rangle \sum_j N_j \alpha_j(\omega),$$

where the angle brackets \(\langle \rangle\) indicate a spatial average over atomic sites. Here

$$\alpha_j(\omega) = \frac{-e^2/m}{\omega^2 - \omega_j^2 + 2i\delta_j\omega}$$

is the atomic polarizability of the Lorentz oscillator that is characterized by \(\omega_j\) and \(\delta_j\) with number density \(N_j\). Furthermore,

$$N = \sum_j N_j$$

is the total number of electrons per unit volume interacting with the applied local electric field.

The electric susceptibility \(\chi_s(\omega)\) of the linear isotropic dielectric medium is defined by the relation
\( P(\omega) = \chi_{e}(\omega)E(\omega), \) (7)

where \( E \) is the macroscopic electric field vector. Under the approximation that \( \langle E_{\text{loc}} \rangle = E \), valid if the number densities \( N_{j} \) are not too large, the electric susceptibility of the medium is found to be

\[
\chi_{e}(\omega) = \sum_{j} N_{j}\chi_{j}(\omega) = -\sum_{j} \frac{N_{j}e^{2}/m}{\omega^{2} - \omega_{j}^{2} + 2i\delta_{j}\omega}.
\] (8)

From Eq. (8) the complex dielectric permittivity \( \varepsilon(\omega) \) of the medium is given as

\[
\varepsilon(\omega) = 1 + 4\pi\chi_{e}(\omega) = 1 - \sum_{j} \frac{b_{j}^{2}}{\omega^{2} - \omega_{j}^{2} + 2i\delta_{j}\omega},
\] (9)

where

\[
b_{j}^{2} = 4\pi N_{j} e^{2} \frac{\omega^{2}}{m}
\] (10)
is the square of the plasma frequency with number density \( N_{j} \). The complex index of refraction \( n(\omega) = n_{r}(\omega) + in_{i}(\omega) \) of the medium is then given by

\[
n(\omega) = [\varepsilon(\omega)]^{1/2} = \left(1 - \sum_{j} \frac{b_{j}^{2}}{\omega^{2} - \omega_{j}^{2} + 2i\delta_{j}\omega}\right)^{1/2}.
\] (11)

The real part of this expression gives the real refractive index \( n_{r}(\omega) \) of the medium, whereas the imaginary part \( n_{i}(\omega) \) is related to the extinction coefficient.

Attention is now given to the differential form of Poynting’s theorem, which may be written as

\[
\nabla \cdot S = -\frac{1}{4\pi} \left( \nabla \cdot H - E \frac{\partial E}{\partial t} + 4\pi \mathbf{P} \frac{\partial \mathbf{P}}{\partial t} \right)
\] (12)
in cgs units. First, consider obtaining an expression for the term \( \mathbf{E} \cdot (\partial \mathbf{P}/\partial t) \) for a double-resonance Lorentz medium that is interacting with a monochromatic plane-wave field. From Eq. (1), we obtain

\[
\mathbf{E} = \frac{m}{e} \left( \frac{d^{2}r_{j}}{dt^{2}} + 2\delta_{j} \frac{dr_{j}}{dt} + \omega_{j}^{2}r_{j} \right)
\]

for \( j = 0, 2 \), and from Eq. (4) the macroscopic polarization vector is found as

\[
\mathbf{P} = -N_{0}e\mathbf{r}_{0} - N_{2}e\mathbf{r}_{2}.
\]

These two expressions then yield

\[
\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} = N_{0}m \left( \frac{d^{2}r_{0}}{dt^{2}} + 2\delta_{0} \frac{dr_{0}}{dt} + \omega_{0}^{2}r_{0} \right) \frac{dr_{0}}{dt}

+ N_{2}m \left( \frac{d^{2}r_{2}}{dt^{2}} + 2\delta_{2} \frac{dr_{2}}{dt} + \omega_{2}^{2}r_{2} \right) \frac{dr_{2}}{dt}

= N_{0}m \left[ \frac{1}{2} \left( \frac{dr_{0}}{dt} \right)^{2} + 2\delta_{0} \left( \frac{dr_{0}}{dt} \right)^{2} + \frac{\omega_{0}^{2}}{2} \frac{d}{dt} (r_{0})^{2} \right]

+ N_{2}m \left[ \frac{1}{2} \left( \frac{dr_{2}}{dt} \right)^{2} + 2\delta_{2} \left( \frac{dr_{2}}{dt} \right)^{2} + \frac{\omega_{2}^{2}}{2} \frac{d}{dt} (r_{2})^{2} \right].
\]

Equation (13) implies that the time-average energy density stored in the double-resonance medium is given by

\[
W_{\text{osc}} = \frac{1}{4} N_{0}m \left[ \left( \frac{dr_{0}}{dt} \right)^{2} + \omega_{0}^{2}(r_{0})^{2} \right]

+ \frac{1}{4} N_{2}m \left[ \left( \frac{dr_{2}}{dt} \right)^{2} + \omega_{2}^{2}(r_{2})^{2} \right].
\] (14)

For a time-harmonic field, Eq. (14) may be written in phasor form so that with Eq. (2) we obtain

\[
W_{\text{osc}} = \frac{1}{4} N_{0}e^{2} \left| E \right|^{2} \frac{\omega^{2} + \omega_{0}^{2}}{(\omega^{2} - \omega_{0}^{2})^{2} + 4\delta_{0}^{2}\omega^{2}}

+ \frac{1}{4} N_{2}e^{2} \left| E \right|^{2} \frac{\omega^{2} + \omega_{2}^{2}}{(\omega^{2} - \omega_{2}^{2})^{2} + 4\delta_{2}^{2}\omega^{2}}.
\] (15)

Finally, substitution of Eq. (10) into Eq. (15) yields

\[
W_{\text{osc}} = \frac{1}{16\pi} \times \left| E \right|^{2} \frac{b_{0}^{2}(\omega^{2} + \omega_{0}^{2})}{(\omega^{2} - \omega_{0}^{2})^{2} + 4\delta_{0}^{2}\omega^{2}} + \frac{b_{2}^{2}(\omega^{2} + \omega_{2}^{2})}{(\omega^{2} - \omega_{2}^{2})^{2} + 4\delta_{2}^{2}\omega^{2}}.
\] (16)

For a completely general Lorentz medium with multiple-resonance frequencies, Eq. (16) generalizes to

\[
W_{\text{osc}} = \frac{1}{16\pi} \left| E \right|^{2} \sum_{j} \frac{b_{j}^{2}(\omega^{2} + \omega_{j}^{2})}{(\omega^{2} - \omega_{j}^{2})^{2} + 4\delta_{j}^{2}\omega^{2}}.
\] (17)

This is the time-average value of the electromagnetic energy density stored in the Lorentz oscillators of the multiple-resonance medium. Furthermore, the time-average value of the energy density stored in the monochromatic plane-wave electromagnetic field is found to be

\[
W_{\text{field}} = \frac{1}{16\pi} \left( n_{r}^{2} + n_{i}^{2} + 1 \right) \left| E \right|^{2}.
\] (18)

Because

\[
n_{r}^{2} - n_{i}^{2} = \varepsilon(\omega) = 1 - \sum_{j} \frac{b_{j}^{2}(\omega^{2} - \omega_{j}^{2})}{(\omega^{2} - \omega_{j}^{2})^{2} + 4\delta_{j}^{2}\omega^{2}},
\] (19)

then the total time-average electromagnetic energy density stored in both the field and the medium is given by

\[
W_{\text{total}} = W_{\text{osc}} + W_{\text{field}}

= \frac{1}{8\pi} \left| E \right|^{2} \left( n_{r}^{2} + \sum_{j} \frac{b_{j}^{2}\omega^{2}}{(\omega^{2} - \omega_{j}^{2})^{2} + 4\delta_{j}^{2}\omega^{2}} \right),
\] (20)

where the summation extends over all the medium resonances.

The energy velocity of a monochromatic field, defined as the rate of electromagnetic energy flow in the medium, is given by the ratio of time-average Poynting vector to the total stored electromagnetic density of the field so that

\[
\mathbf{v}_{E} = \frac{\mathbf{S}}{W_{\text{total}}}. \quad (21)
\]
For a monochromatic plane-wave field, the time-average value of the magnitude of the Poynting vector is readily found to be
\[
\langle S \rangle = \frac{c}{8\pi} n_r |E|^2,
\]
where \(c\) is the speed of light in vacuum. When Eqs. (20)–(22) are used, the magnitude of the energy-transport velocity in a multiple-resonance Lorentz medium is found to be
\[
v_E = \frac{c}{n_r(\omega) + \frac{1}{n_r(\omega)} \sum_j \omega_j^2 \omega_b^2 \left( \omega^2 - \omega_j^2 \right)^2 + 4\delta_j^2 \omega^2}.
\]
This important result is the desired generalization of the well-known expression for the energy velocity in a single-resonance Lorentz medium according to Loudon\(^2\) and reduces to his expression in that special case.

**DISCUSSION**

The total time-average energy density given in Eq. (20) includes the time-average electric field energy density
\[
W_E = \frac{1}{16\pi} |E|^2
\]
and the time-average magnetic field energy density
\[
W_H = \frac{1}{16\pi} |\mathbf{H}|^2 = \frac{1}{16\pi} (n_r^2 + n_i^2) |E|^2
\]
as well as what may be defined as the potential and kinetic energy densities of the Lorentz oscillators, given, respectively, by
\[
W_P = \frac{1}{16\pi} |E|^2 \sum_j \omega_j^2 \frac{b_j^2 \omega_j^2}{\left( \omega^2 - \omega_j^2 \right)^2 + 4\delta_j^2 \omega^2},
\]
\[
W_K = \frac{1}{16\pi} |E|^2 \sum_j \omega_j^2 \frac{b_j^2 \omega_j^2}{\left( \omega^2 - \omega_j^2 \right)^2 + 4\delta_j^2 \omega^2}.
\]
Comparison of Eq. (24d) with the denominator in Eq. (23) shows that the energy velocity may be expressed as
\[
v_E = \frac{c}{n_r(\omega) + \frac{1}{n_r(\omega)} \frac{W_K}{W_E} \frac{1}{n_r(\omega) W_E}} \frac{W_K}{W_E} = \frac{c}{n_r(\omega) + \frac{1}{n_r(\omega)} \frac{W_K}{W_E} \frac{1}{n_r(\omega) W_E}} \frac{W_K}{W_E},
\]
with the use of Eq. (24a). The quantity \(W_K/W_E\) is the ratio of the time-average kinetic energy density of the Lorentz oscillators with respect to the time-average electric field energy density.

For a single-resonance Lorentz medium, the ratio \(W_K/W_E\) is given by
\[
\frac{W_K}{W_E} = \frac{b_0^2 \omega_0^2}{\left( \omega^2 - \omega_0^2 \right)^2 + 4\delta_0^2 \omega^2},
\]
which may be rewritten in the form
\[
\frac{W_K}{W_E} = \frac{(\epsilon_r - \epsilon_0)(\epsilon_r - \epsilon_\infty) + \epsilon_i^2}{\epsilon_0 - \epsilon_\infty}.
\]

Here, \(\epsilon_r\) is the real part and \(\epsilon_i\) is the imaginary part of the complex dielectric permittivity
\[
\epsilon(\omega) = 1 - \frac{b_0^2}{\omega^2 - \omega_0^2 + 2i\delta_0 \omega},
\]
where
\[
\epsilon_0 = \epsilon(0),
\]
\[
\epsilon_\infty = \epsilon(\infty)
\]
are the zero-frequency and infinite-frequency values, respectively, of the complex dielectric permittivity. With Eq. 25, the energy velocity may be expressed as
\[
\frac{W_K}{W_E} = \frac{b_0^2 \omega_0^2}{\left( \omega^2 - \omega_0^2 \right)^2 + 4\delta_0^2 \omega^2},
\]
which reduces to his expression in that special case.
expression (25) for the energy-transport velocity becomes

\[ v_E = \frac{c}{n_r(\omega) + \frac{[\epsilon_r(\omega) - \epsilon_0][\epsilon_r(\omega) - \epsilon_\infty] + \epsilon_\infty^2(\omega)}{n_r(\omega)(\epsilon_0 - \epsilon_\infty)}}. \]  

The advantage of this equation is that it is in a form that is independent of the medium model and so is likely to be applicable to other types of dispersive media. It is also a good approximation of the energy velocity in a multiple-resonance medium for signal frequencies that are in the vicinity of the single-resonance frequency modeled. In that case, \( \epsilon_\infty \) is the apparent background permittivity below the single-resonance frequency considered, and \( \epsilon_\infty \) is the apparent background permittivity above the resonance frequency.

The behavior of the energy-transport velocity in a double-resonance Lorentz medium as a function of the applied signal frequency, as described by either Eq. (23) or Eq. (25), is depicted by the solid curves in Figs. 1 and 2. The medium parameters for Fig. 1 are \( \omega_0 = 1 \times 10^{16} \text{sec}^{-1}, b_0^2 = 5 \times 10^{32} \text{sec}^{-2}, \delta_0 = 0.1 \times 10^{16} \text{sec}^{-1}, \omega_2 = 4 \times 10^{16} \text{sec}^{-1}, b_2^2 = 20 \times 10^{32} \text{sec}^{-2}, \) and \( \delta_2 = 0.28 \times 10^{16} \text{sec}^{-1}; \) in Fig. 2 only the second-resonance frequency has been changed to \( \omega_2 = 7 \times 10^{16} \text{sec}^{-1}. \) This increase in the second-resonance frequency of the medium not only shifts the location of the second minimum in the energy velocity to higher frequencies but also shifts the location of the maximum in the energy velocity that occurs between the two resonance frequencies to a higher frequency as well as increases the value of this relative maximum. The dotted curve in each figure depicts the normalized ratio \( W_K/W_E \) of the time-average kinetic energy density of the Lorentz oscillators with respect to the time-average electric field energy density. It is then clear that the energy velocity is a minimum in the frequency regions where the kinetic energy of the Lorentz oscillators is a maximum.

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