Numerical determination of the signal velocity in dispersive pulse propagation

Kurt E. Oughstun, Philippe Wyns, and Daniel Foty*

Department of Computer Science and Electrical Engineering, University of Vermont, Burlington, Vermont
05403

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The signal velocity of a unit-step-function-modulated signal with a constant carrier frequency \( \omega_c \) that is propagating in a linear dispersive medium with absorption (the Lorentz medium) is determined through numerical simulation and compared with that predicted by rigorous asymptotic theory. The exceptional agreement between these purely numerical results and the asymptotic theory, including the bifurcation of the signal velocity at carrier frequencies well above the medium resonance frequency, serve to validate completely both the description of the signal arrival afforded by the asymptotic theory and the physical propriety of this velocity measure in dispersive pulse propagation for input pulses with an instantaneous turn-on time.

1. INTRODUCTION

A quantity of fundamental importance to the physical description of dispersive pulse propagation in a causal medium is the signal velocity. This important velocity measure was first introduced by Sommerfeld\(^1\) and Brillouin\(^2,3\) in 1914, using the asymptotic method of steepest descents to describe the propagation of a unit-step-function-modulated signal of constant carrier frequency \( \omega_c \) in a single-resonance Lorentz medium. Because of the unnecessary constraint imposed on the deformed contour of integration in their analysis, the signal arrival was defined to occur when the path of steepest descent crossed the simple pole singularity at \( \omega = \omega_0 \) in the initial envelope spectrum. This then resulted in a frequency dependence of the signal velocity that erroneously peaks to the vacuum speed of light \( c \) near the resonance frequency \( \omega_0 \) of the medium and that is incomplete in its description for applied signal frequencies above \( \omega_0 \). This was corrected in part by Baerwald,\(^4\) who showed in 1930 that the signal velocity reaches a minimum near the resonance frequency; these results were also presented by Brillouin in his book.\(^3\)

In a more recent consideration of the signal velocity,\(^5\) it was concluded that the measurable signal velocity should lie somewhere between the values predicted by Brillouin and Baerwald. However, no clear physical interpretation of the signal arrival is provided by these descriptions. In a lossless medium with dispersion the signal arrives appears in the field evolution as a pronounced increase in the field amplitude to a level above that of the precursor fields, much in the same way that the field amplitude in Fresnel diffraction phenomena increases when the observation point is moved from the geometrical shadow to the geometrical region of illumination.\(^6-10\) The more-recent calculations of Barakat\(^11\) were focused on the signal properties of ultrashort-pulse propagation in a dispersive medium without absorption, with similar results. Such is not the case in a causal dispersive medium, in which the precursor fields may easily be comparable in amplitude with that of the steady-state field. Because of the ambiguity in physically measuring the precise point in the field evolution at which such a transition is made from the precursor field evolution to the signal evolution,\(^5\) the signal velocity has been criticized by some authors\(^12,13\) as either not being a useful physical quantity or simply not agreeing with their experimental observations.\(^14\)

The recent analysis\(^5,16,17\) of linear dispersive pulse propagation in a Lorentz medium by using modern asymptotic techniques resulted in significant quantitative improvements in the entire description of the dynamical field evolution and, in addition, yielded precise, correct definitions of the signal arrival and the resultant signal velocity. This asymptotic description of the propagated field relies on Olver’s saddle-point method,\(^18\) in which the deformed contour of integration no longer is constrained to lie along the path of steepest descent through the relevant saddle points. Because of this result, the signal arrival was redefined to occur when a transition in asymptotic dominance was made between the dominant saddle point and the residue of the simple pole singularity at \( \omega = \omega_c \). It was found that the resultant frequency dependence of the signal velocity then reaches a minimum level slightly above the resonance frequency of the single-resonance Lorentz medium, in agreement with Baerwald’s result, and also bifurcates into an anterior presignal, a posterior presignal, and a main signal velocity for applied carrier frequencies \( \omega_c > \omega_{SB} \), where \( \omega_{SB} > \omega_0 \). In addition, the frequency dependence of the energy-transport velocity for a time-harmonic field of frequency \( \omega_c \) in a single-resonance Lorentz medium, as derived by Loudon,\(^19\) was found to be in excellent agreement with the upper branches of the signal-velocity curves for applied signal frequencies in either of the two frequency intervals \( 0 \leq \omega_c < \omega_0 \) and \( \omega_0 > \omega_{SB} \).\(^16,17\) Similar agreement was also obtained for the case of a double-resonance Lorentz medium.\(^20,21\) This direct correlation between the signal velocity of a pulsed field and the energy-transport velocity of a time-harmonic field in a Lorentz medium has led to a new physical model of linear dispersive pulse propagation\(^22\) that supplants the previous group-velocity description, which becomes inadequate in cases of dispersive media with absorption.

In the present paper we focus on a numerical experiment...
for the determination of the signal velocity for a unit-step-
function-modulated signal of carrier frequency \( \omega_c \) that is
propagating in a single-resonance Lorentz medium. The
signal velocity is determined here solely from the calculated
propagated field structure through the numerically mea-
sured instantaneous angular frequency of oscillation of the
field. This physical measure of the signal velocity, together
with its complete agreement with the frequency behavior
predicted by the modern asymptotic theory, serves to prove
the physical propriety of this velocity measure in linear-
dispersive-pulse-propagation phenomena.

2. PHYSICAL MEASURE OF THE SIGNAL VELOCITY

In a single-resonance Lorentz medium with complex index
of refraction

\[ n(\omega) = \left( 1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta_0} \right)^{1/2}, \]

(2.1)

the propagated plane-wave field that is due to an input unit-
step-function-modulated signal with carrier frequency \( \omega_c \) is
given by the integral representation\(^\text{1,7}\)

\[ A(z, t) = -\frac{1}{2\pi} \text{Re} \left\{ \int_{\omega - \omega_c}^{\omega + \omega_c} \exp \left[ i \phi(\omega, \theta) \right] \omega d\omega \right\}, \]

(2.2)

with the complex phase function

\[ \phi(\omega, \theta) = i\omega[n(\omega) - \theta]. \]

(2.3)

Here

\[ \theta = \frac{ct}{z}, \]

(2.4)

is the time parameter that characterizes the dynamical
field evolution, in which \( z \) is the propagation distance, \( t \)
is the time, and \( c \) is the vacuum speed of light. For the single-
resonance Lorentz medium with a refractive index described
by Eq. (2.1), the complex phase function Eq. (2.3) possesses
two sets of saddle points: two near saddle points whose real
coordinate position is, in absolute value, located in the low-
frequency region below the medium resonance frequency \( \omega_0 \)
and two distant saddle points whose real coordinate position is,
in absolute value, located in the high-frequency region above
the frequency \( \omega_1 = (\omega_0^2 + b^2)^{1/2} \). Notice that the
absorption band of the medium is located approximately
between these two frequency values.

The integral representation [Eq. (2.2)] vanishes identically
for \( \theta < 1 \) so that the total field propagates with a velocity
less than the vacuum speed of light.\(^1,3,7\) The asymptotic
approximation of the propagated field for \( \theta \geq 1 \) may be
written in the form\(^7\)

\[ A(z, t) \approx A_0(z, t) + A_S(z, t) + A_\theta(z, t) \]

(2.5)
as \( z \to \infty \). Here \( A_0(z, t) \) and \( A_\theta(z, t) \) are negligible when the
first or Sommerfeld precursor field is predominant, \( A_S(z, t) \)
and \( A_\theta(z, t) \) are negligible when the second or Brillouin
precursor field is predominant, and \( A_0(z, t) \) and \( A_S(z, t) \) are
negligible when the signal contribution is predominant.
Two of the terms become important at the same time during
periods of transition, giving a continuous asymptotic de-
scription of the time evolution of the total propagated field
for large \( z \). The dynamical field evolution that is due to an
input unit-step-function-modulated signal is found to de-
pend on whether the carrier frequency \( \omega_c \) is greater or less
than the real frequency value.\(^6,17\)

\[ \omega_{SB} \approx \omega_0 \left( 2 + \frac{b^2}{\omega_0^2} + \frac{5b^2}{3\omega_0^2} \right)^{1/2}. \]

(2.6)
The precise value of \( \omega_{SB} \) is given by the real frequency
coordinate of the distant saddle point in the right-hand half of
the complex \( \omega \) plane at \( \theta = \theta_{SB} \) when the distant and upper
near saddle points have equal dominance, for which

\[ \theta_{SB} \approx \theta_0 - \frac{4\delta^2b^2}{\theta_0^4} \left[ \frac{27\delta^2\delta^4(\theta_0 - 1)^2}{4\theta_0^4} \right]^{1/3} \times \left[ \left( 1 + \frac{b^2}{2\theta_0(\theta_0 - 1)\omega_0^4} \right)^{1/2} + 1 \right]^{1/3} \]

\[ \left[ \left( 1 + \frac{b^2}{2\theta_0(\theta_0 - 1)\omega_0^4} \right)^{1/2} - 1 \right]^{1/3} \]

(2.7)
is a fairly accurate approximation for this \( \theta \) value.

Consider first the dynamical evolution of the field as a
function of the space–time parameter \( \theta \) when \( \omega_c < \omega_{SB} \).
Initially the distant saddle points are the dominant contri-
bution to the asymptotic behavior of the propagated field for
\( 1 \leq \theta < \theta_{SB} \) and yield the Sommerfeld precursor evolution.
Because these saddle points are located at infinity just below
the real frequency axis when \( \theta = 1 \) and approach the outer
branch points of the complex index of refraction [Eq. (2.1)]
as \( \theta \to \infty \), the instantaneous angular frequency of oscillation
of the Sommerfeld precursor decreases monotonically from
infinity at \( \theta = 1 \) and approaches the value \( (\omega_0 - \delta)^{1/2} \)
 asymptotically from above as \( \theta \) increases indefinitely.\(^7\)
Over the space–time interval \( \theta_{SB} < \theta < \theta_1 \), for which

\[ \theta_1 \approx \theta_0 + \frac{2\delta^2b^2}{\theta_0(3\omega_0^2 - 4\delta^2)}, \]

(2.8)

\[ \theta_0 = n(0) = \left( 1 + \frac{b^2}{\omega_0^2} \right)^{1/2}, \]

(2.9)

\[ \alpha = 1 - \frac{b^2}{3\omega_0^2\omega_1^2} (4\omega_1^2 + b^2), \]

(2.10)

the upper near saddle point along the imaginary frequency
axis is the dominant contribution, and over the space–time
interval \( \theta_1 \leq \theta < \theta_c \) the two near saddle points are the
dominant contributions to the asymptotic behavior of the
propagated field and yield the Brillouin precursor evolution.
Because the upper near saddle point is situated initially
along the positive imaginary axis, is at the origin at \( \theta = \theta_0 \),
descends along the negative imaginary axis for \( \theta_0 < \theta < \theta_1 \),
and coalesces with the lower near saddle point when \( \theta = \theta_1 \),
after which both near saddle points symmetrically move off
the imaginary axis and approach the inner branch points of
Eq. (2.1) as \( \theta \to \infty \), the instantaneous angular frequency of
field. Over this interval the main signal evolves with a fixed angular frequency of oscillation equal to the input carrier frequency \( \omega_c \) of the unit-step-function-modulated signal.

The precise value of \( \theta_c \) is defined by the equation\(^{10,17}\)

\[
X(\omega_{SP}, \theta_c) = X(\omega_c), \quad \theta_c < \theta_{SP},
\]

where \( X(\omega, \theta) = \text{Re}[\phi(\omega, \theta)] \). Notice that the behavior of \( X(\omega, \theta) \) is independent of \( \theta \) along the real frequency axis. The velocity at which this transition point in the field evolution propagates is then given by

\[
v_c = \frac{c}{\theta_c},
\]

and is called the main signal velocity. This point is characterized physically in the total field evolution by the instantaneous frequency of oscillation. For \( \theta < \theta_c \) the field is dominated either by the Sommerfeld precursor whose instantaneous angular frequency \( \omega_{SP} \) satisfies the inequality \( \omega_c \geq \omega_{SP} \) or by the Brillouin precursor whose instantaneous angular frequency \( \omega_B \) either approaches \( \omega_c \) from below as \( \theta \) approaches \( \theta_c \) from below, if \( \omega_c = (\omega_0^2 - \delta^2)^{1/2} \), or is bounded below \( \omega_c \) for all \( \theta > \theta_c \). Hence it is only for \( \theta > \theta_c \) that the field oscillates predominantly at the input carrier frequency \( \omega_c \) of the signal. The value of \( \theta_c \) may therefore be measured numerically (experimentally) by determining the space–time point at which the field first begins to oscillate physically at \( \omega = \omega_c \).

Consider finally the dynamical field evolution for \( \omega_c > \omega_{SB} \). The total field evolution then begins with the Sommerfeld precursor, which dominates the field evolution over the space–time interval \( 1 \leq \theta < \theta_{C1} \). At \( \theta = \theta_{C1} \), where\(^{17}\)

\[
X(\omega_{SP}, \theta_{C1}) = X(\omega_c), \quad 1 < \theta_{C1} < \theta_{SB},
\]

the distant saddle points are of equal asymptotic dominance with the simple pole singularity at \( \omega = \omega_c \). This pole contribution is the dominant contribution to the asymptotic behavior of the propagated field over the space–time interval \( \theta_{C1} < \theta < \theta_{C2} \). At \( \theta = \theta_{C2} \), where\(^{17}\)

\[
X(\omega_{SP}, \theta_{C2}) = X(\omega_c), \quad \theta_{SB} < \theta_{C2} < \theta_0,
\]

the pole contribution is of equal asymptotic dominance with the upper near saddle point along the imaginary frequency axis. The Brillouin precursor then dominates the field evolution over the space–time interval \( \theta_{C2} < \theta < \theta_c \). At \( \theta = \theta_{C2} \), for which \( \theta_c \) is given by Eq. (2.11), the near saddle points are of equal asymptotic dominance with the simple pole singularity at \( \omega = \omega_c \). This pole contribution is then the dominant contribution to the asymptotic behavior of the propagated field for all \( \theta > \theta_c \).

The velocity at which the transition point at \( \theta = \theta_{C1} \) in the field evolution propagates is given by

\[
v_{C1} = \frac{c}{\theta_{C1}}, \quad \omega_c > \omega_{SB}
\]

and is called the anterior presignal velocity; the velocity at which the transition point at \( \theta = \theta_{C2} \) in the field evolution propagates is given by

\[
v_{C2} = \frac{c}{\theta_{C2}}, \quad \omega_c > \omega_{SB}
\]

and is called the posterior presignal velocity; and the veloci-

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**Fig. 1.** Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency \( \omega_c = 1 \times 10^3 \text{sec} \) at the propagation distances (a) \( z = 1 \times 10^{-6} \) cm, (b) \( z = 1 \times 10^{-4} \) cm, and (c) \( z = 1 \times 10^{-5} \) cm. Notice the change in horizontal scale in (c).
Fig. 2. Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 1.

Fig. 3. Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency $\omega_c = 3.0 \times 10^{10} \text{sec}^{-1}$ at the propagation distances (a) $z = 1 \times 10^{-6} \text{cm}$, (b) $z = 5 \times 10^{-6} \text{cm}$, and (c) $z = 1 \times 10^{-4} \text{cm}$.

Fig. 4. Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 3.

Fig. 5. Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency $\omega_c = 4.0 \times 10^{10} \text{sec}^{-1}$ at the propagation distances (a) $z = 5 \times 10^{-7} \text{cm}$, (b) $z = 1 \times 10^{-6} \text{cm}$, and (c) $z = 5 \times 10^{-6} \text{cm}$. 
Fig. 6. Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 5.

Fig. 8. Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 7.

Fig. 7. Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency $\omega_c = 4.4 \times 10^{16}$/sec at the propagation distances (a) $z = 5 \times 10^{-7}$ cm, (b) $z = 1 \times 10^{-6}$ cm, and (c) $z = 5 \times 10^{-6}$ cm.

Fig. 9. Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency $\omega_c = 4.8 \times 10^{16}$/sec at the propagation distances (a) $z = 5 \times 10^{-7}$ cm, (b) $z = 1 \times 10^{-6}$ cm, and (c) $z = 5 \times 10^{-6}$ cm.
Fig. 10. Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 9.

Fig. 11. Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency \( \omega_c = 5.5 \times 10^6 \) rad/sec at the propagation distances (a) \( z = 5 \times 10^{-7} \) cm, (b) \( z = 1 \times 10^{-6} \) cm, and (c) \( z = 5 \times 10^{-6} \) cm.

The value of \( \theta_{c1} \) may be numerically (experimentally) measured by determining the space–time point at which the field first begins to oscillate physically at \( \omega = \omega_{c1} \). The value of \( \theta_{c2} \) may be measured by determining the subsequent space–time point at which the field first ceases to oscillate physically at \( \omega = \omega_{c2} \). The values of \( \theta_{c1} > \theta_{c2} \) may be determined by measuring the phase shift between the field at the point \( \theta_{c1} \) and the phase-shifted field at the point \( \theta_{c2} \).

3. NUMERICAL RESULTS

The instantaneous angular frequency of oscillation of the propagated field structure that is due to an input unit-step-function-modulated signal of carrier frequency \( \omega_c \) may be determined directly from the numerically determined field evolution at a fixed propagation distance \( z \) when the frequency is considered a function of the space–time parameter \( \theta = ct/z \). The dynamical field evolution at fixed \( z \) and \( \omega_c \) is calculated here by using the numerical procedure for inverting Laplace-transform-type integrals that is described in a companion paper.23 The results are displayed most conveniently as a function of \( \theta \), since critical aspects of the field evolution are then independent of the propagation distance.

Let \( \Delta \theta_i = \theta_{i+1} - \theta_i \) represent the absolute difference between the \( \theta \) values at successive zero crossings in the calculated field evolution at a given value of the propagation distance. The associated period for a monochromatic field with this zero-crossing spacing is then given by

\[
T_i = \frac{2\pi}{\Delta \theta_i},
\]

and the corresponding angular frequency of oscillation is

\[
\omega_i = \frac{2\pi}{T_i}.
\]

The results are displayed most conveniently as a function of \( \theta \), since critical aspects of the field evolution are then independent of the propagation distance.

Let \( A_{0j} = 0_j + 0_j \) represent the absolute difference between the \( 0 \) values at successive zero crossings in the calculated field evolution at a given value of the propagation distance. The associated period for a monochromatic field with this zero-crossing spacing is then given by

\[
T_j = \frac{2\pi}{\Delta \theta_j},
\]

and the corresponding angular frequency of oscillation is

\[
\omega_j = \frac{2\pi}{T_j}.
\]

The value of \( \theta_{c1} \) may be numerically (experimentally) measured by determining the space–time point at which the field first begins to oscillate physically at \( \omega = \omega_{c1} \). The value of \( \theta_{c2} \) may be measured by determining the subsequent space–time point at which the field first ceases to oscillate physically at \( \omega = \omega_{c2} \). The values of \( \theta_{c1} > \theta_{c2} \) may be determined by measuring the phase shift between the field at the point \( \theta_{c1} \) and the phase-shifted field at the point \( \theta_{c2} \).
Fig. 12. Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 11.

Fig. 13. Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency $\omega_c = 6.5 \times 10^{16}$ sec at the propagation distances (a) $z = 1 \times 10^{-5}$ cm, (b) $z = 5 \times 10^{-5}$ cm, and (c) $z = 1 \times 10^{-4}$ cm.

Fig. 14. Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 13.

Fig. 15. Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency $\omega_c = 8.5 \times 10^{16}$ sec at the propagation distances (a) $z = 1 \times 10^{-5}$ cm, (b) $z = 5 \times 10^{-5}$ cm, and (c) $z = 1 \times 10^{-4}$ cm.
If the instantaneous angular frequency of oscillation for a nonmonochromatic field with zero dc bias is monotonically decreasing, increasing, or fixed over some $\theta$ interval, then the frequency value given by Eq. (3.2) is assumed by the field at some $\theta$ value in the interval $(\theta_i, \theta_{i+1})$. Such is the case in the evolution of the individual Sommerfeld and Brillouin precursors as well as in the signal evolution that is due to the pole contribution in the dynamical field evolution of a unit-step-function-modulated signal in a single-resonance Lorentz medium, as considered here. For convenience, the frequency value of Eq. (3.2) is assigned here to the middle of the interval $(\theta_i, \theta_{i+1})$. With the exception of the lowest-frequency structure present in the Brillouin precursor, this arbitrary assignment results in only an extremely small error in the frequency evolution of the instantaneous angular frequency of the propagated field structure.

The transition points $\theta_o$, $\theta_{o1}$, and $\theta_{o2}$ may then be obtained directly from the $\theta$ evolution of the numerically obtained instantaneous angular frequency of oscillation. For $\omega_o < \omega_{SB}$ the value of $\theta_o$ is obtained when the frequency first stabilizes at the input carrier frequency $\omega_o$. For $\omega_o > \omega_{SB}$ the value of $\theta_{o1}$ is obtained when the frequency first stabilizes at $\omega_o$, in the prepulse evolution, the value of $\theta_{o2}$ is obtained when the frequency first departs from $\omega_o$ in the prepulse evolution, and the value of $\theta_o$ is obtained when the frequency again stabilizes at $\omega_o$. Because of unavoidable numerical errors that occur in the calculation of the dynamical field evolution and the subsequent calculation of the instantaneous oscillation frequency, the determinations of these transition points are performed to within a small frequency uncertainty $\Delta\omega_o$ greater than and less than the carrier frequency $\omega_o$. This then results in an error in the numerically determined values of $\theta_o$, $\theta_{o1}$, and $\theta_{o2}$ which is indicated in the results.

The calculations presented in this paper were performed for a single-resonance Lorentz medium with parameters

$$\omega_0 = 4.0 \times 10^{16} / \text{sec},$$
$$b^2 = 20.0 \times 10^{22} / \text{sec}^2,$$
$$\delta = 0.14 \times 10^{16} / \text{sec},$$

which correspond to a highly absorptive medium. The corresponding frequency and $\theta$ values given by relations (2.6)–(2.9) are then

$$\omega_{SB} \approx 7.21 \times 10^{16} / \text{sec} \quad \text{(actual value = 8.6 \times 10^{16} / \text{sec}),}$$
$$\theta_{SB} \approx 1.295 \quad \text{(actual value = 1.35),}$$
$$\theta_1 \approx 1.501,$$
$$\theta_0 = 1.5.$$

The actual (numerically determined) values are $\theta_{SB} = 1.35$ and $\omega_{SB} = 8.6 \times 10^{16} / \text{sec}$, as indicated in the parentheses above. The absorption band of the medium extends (approximately) from $(\omega_o^2 - b^2)^{1/2} = 3.998 \times 10^{16} / \text{sec}$ to $(\omega_o^2 - b^2)^{1/2} = 5.998 \times 10^{16} / \text{sec}$, where $\omega_o^2 = \omega_0^2 + b^2$.

The results of this numerical study are depicted in Figs. 1–20 for several increasing values of the input carrier frequency $\omega_o$. Each odd-numbered figure of this set depicts the numerically determined dynamical field evolution as a function of the space–time parameter $\theta$ for several values of the propagation distance $z$ at a fixed value of $\omega_o$. Each even-numbered figure of this set depicts the $\theta$ evolution of the
instantaneous angular frequency of oscillation for the dynamical field evolution in the preceding figure. In the even-numbered subset of Figs. 2–14 the 's indicate the numerical values for the smallest propagation distance, the 's indicate the numerical values for the middle propagation distance, and the 's indicate the numerical values for the largest propagation distance of the field evolution in the previous figure. In Figs. 16, 18, and 20 the 's and 's have been interchanged. The solid horizontal line in each even-numbered figure indicates the value of the input carrier frequency \( \omega_c \). The resultant numerically determined values of \( \theta_{SB}, \theta_{Cl}, \) and \( \theta_{S} \) are indicated in these figures (when appropriate) along with their associated ambiguity resulting from both numerical error and the permitted frequency uncertainty about the value of \( \omega_c \).

In Figs. 1–4 the applied signal frequency \( \omega_c \) is below the undamped resonance frequency \( \omega_R \) of the medium and hence is below the absorption band of the medium. The dynamical field evolution then begins with the Sommerfeld precursor evolution over \( 1 \leq \theta < \theta_{SB} \), followed by the Brillouin precursor, which evolves over \( \theta_{SB} \leq \theta < \theta_{Cl} \), and then by the main signal evolution over \( \theta > \theta_{Cl} \), in complete agreement with the modern asymptotic description.\(^{15,17}\) Because of the low carrier frequency used in Fig. 1, the Sommerfeld precursor amplitude is nearly negligible in comparison with both the Brillouin precursor and main signal amplitudes and is barely visible in Fig. 1(b). Because of this, these calculations were not focused on ensuring that the Sommerfeld precursor evolution was always sampled properly; for example, it is sampled sufficiently in Fig. 3(b) but not in Fig. 3(c). This undersampling is reflected in the \( \theta \) evolution of the instantaneous angular frequency of oscillation for the Sommerfeld precursor in Figs. 2 and 4, but it does not affect the determination of the main signal arrival point \( \theta_c \) (notice that there is no prepulse evolution, since \( \omega_c < \omega_{SB} \) here). As shown in Figs. 2 and 4, the instantaneous oscillation frequency of the Brillouin precursor increases monotonically from zero such that the oscillation frequency of the total field reaches \( \omega_c \) from below and remains fixed at \( \omega_c \) for all \( \theta > \theta_c \).

In passing it is worthwhile to point out that the propagation distance in Fig. 1(a) is sufficiently small that the propagated field evolution depicted is in the immature dispersion regime. This is reflected in its associated frequency evolution in Fig. 2, which is significantly different from that for the field evolution in Figs. 1(b) and 1(c). The propagation distance in the latter two figures is sufficiently large for the propagated field evolution to be in the mature dispersion limit for which the asymptotic theory applies.

The calculations illustrated in Figs. 5–12 are for input carrier frequencies that increase up through the absorption band of the medium. Again, the signal arrival is preceded by the Brillouin precursor, and the instantaneous angular frequency of oscillation of the field approaches \( \omega_c \) from below and remains fixed at \( \omega_c \) for all \( \theta > \theta_c \). Notice that the onset and the initial evolution of the Sommerfeld precursor field are not readily visible in these figures because of the large range in values of \( \theta \) necessary for reaching the steady-state field evolution. As in the previous set of figures, there is no prepulse evolution, since \( \omega_c < \omega_{SB} \) in this set of calculations. Because of the large medium attenuation at these carrier frequencies, the field calculations at the largest propagation distance considered here result in a main signal.

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**Fig. 18.** Evolution of the instantaneous angular frequency of oscillation for the dynamical field evolution of Fig. 17.

**Fig. 19.** Dynamical field evolution of an input unit-step-function-modulated signal with carrier frequency \( \omega_c = 10.0 \times 10^9 / \text{sec} \) at the propagation distances (a) \( z = 1 \times 10^{-3} \) cm, (b) \( z = 5 \times 10^{-5} \) cm, and (c) \( z = 1 \times 10^{-4} \) cm.
amplitude that is nearly zero, as shown in Figs. 5(c), 7(c), and 9(c). The corresponding calculations of the instantaneous oscillation frequency are then somewhat inaccurate, particularly in Figs. 6 and 8, in which the absorption at \( \omega_c \) is near its maximum value. However, this does not affect the determination of \( \theta_c \) since an accurate description of the frequency evolution is obtained for the smaller propagation distances.

Notice that the field evolution depicted in Figs. 5, 7, 9, and 11 is dominated by the precursor evolution, particularly at the larger propagation distances, since the associated attenuation is much less than that at the input carrier frequency. For applied signal frequencies at the lower end of the absorption band (\( \omega_c \approx \omega_0 \)) the dominant field structure at large propagation distances is dominated by the Brillouin precursor, as seen in Fig. 5. As the input signal frequency increases up through the absorption band to \( \omega_1 \) the trailing edge of the Sommerfeld precursor becomes more noticeable as it interferes increasingly with the Brillouin precursor evolution. Because the propagation distances are so small, the precursor evolution depicted here is primarily in the immature dispersion regime. It is clear that an experimental measurement of this propagated field structure would detect only the precursor field (unless the propagation distance were exceedingly small) and consequently would measure a field velocity that is associated with the observed peak in the precursor field. This velocity would be close to the vacuum speed of light, which is much greater than the proper signal velocity.

As the applied signal frequency \( \omega_c \) is increased to levels above the medium absorption band, the Sommerfeld precursor field becomes more pronounced in the total field evolution, as is readily evident in Fig. 13. In Fig. 13(a) the propagation distance is sufficiently small to yield a propagated field structure that is in the immature dispersion limit, wherein the precursor field structure is not yet well defined, whereas in Fig. 13(c) it is large enough to be in the mature dispersion limit wherein the precursor field structure is clearly well defined. The propagated field structure is between these two limits in Fig. 13(b), in which the interference between the trailing edge of the Sommerfeld precursor and the Brillouin precursor is clearly evident. Although this interference distorts the behavior of the instantaneous oscillation frequency of the propagated field structure, as shown in Fig. 14, a well-defined signal arrival at \( \theta = \theta_c \) is obtained readily, as indicated in the figure. It is clear from these diagrams that there is no prepulse formation at this value of \( \omega_c \) (which is still less than \( \omega_{SB} \)). Notice that the medium attenuation at \( \omega_c \) is still much larger than that over much of the precursor field evolution, so that for the largest propagation distance considered [Fig. 13(c)] the amplitude of the main signal evolution is nearly zero. As in the previous set of calculations, this results in an inaccurate evolution of the oscillation frequency, as seen in Fig. 14.

For the dynamical field evolution depicted in Fig. 15 the input signal frequency \( \omega_c = 8.5 \times 10^{16}/\sec \) is slightly less than the value \( \omega_{SB} = 8.7 \times 10^{16}/\sec \). Figure 16 shows that the instantaneous angular frequency of oscillation of the Sommerfeld precursor approaches \( \omega_c \) from above but does not reach this value because of the subsequent evolution of the Brillouin precursor field. Thus no prepulse evolution is observed here, in agreement with the modern asymptotic theory.\(^{17}\) Because of the interference between the trailing edge of the Sommerfeld precursor and the Brillouin precursor field, the instantaneous oscillation frequency takes values within a range above and below \( \omega_c \) and finally stabilizes at \( \omega_c \) for \( \theta = \theta_c \) as indicated in the figure. The dynamical field and associated frequency evolution of the propagated field for several values of the input carrier frequency \( \omega_c \) increasing above \( \omega_{SB} \) are illustrated in Figs. 17–20. Figures 18 and 20 show that the instantaneous oscillation frequency first reaches \( \omega_c \) at \( \theta = \theta_{cl} \), is equal to \( \omega_c \) for \( \theta_{cl} < \theta < \theta_{c2} \), then takes values greater than or less than \( \omega_c \) when \( \theta \) increases above \( \theta_{c2} \) (because of interference with the Brillouin precursor evolution), and finally stabilizes at \( \omega_c \) at \( \theta = \theta_c \) and remains at that value thereafter. The prepulse formation therefore is clearly obtained in these cases, in complete agreement with the modern asymptotic theory.\(^{10,17}\)

The numerically determined values of \( \theta_{cl}, \theta_{c2} \), and \( \theta_c \) at each value of \( \omega_c \) for these and additional field calculations are given in Table 1 along with their associated margins of error. The resultant relative signal-velocity values \( v_{cl/c} = 1/v_{cl}, v_{c2/c} = 1/v_{c2}, \) and \( v_c/c = 1/\theta_c \) are illustrated in Fig. 21 as a function of the input carrier frequency \( \omega_c \). The solid

<table>
<thead>
<tr>
<th>( \omega_c , (10^{16}/\sec) )</th>
<th>( \theta_{cl} )</th>
<th>( \theta_{c2} )</th>
<th>( \theta_c )</th>
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<tr>
<td>1.0</td>
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<td>1.44 ± 0.05</td>
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*Present address, General Technology Division, IBM Corporation, Essex Junction, Vermont 05452.

REFERENCES