Pulse centroid velocity of the Poynting vector

Natalie A. Cartwright and Kurt E. Oughstun

College of Engineering and Mathematics, University of Vermont, Burlington, Vermont 05405

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The evolution of the pulse centroid velocity of the Poynting vector for both ultrawideband rectangular and ultrashort Gaussian envelope pulses is presented as a function of the propagation distance in a dispersive, absorptive dielectric material. The index of refraction of the material is described by the Lorentz–Lorenz formula in which a single-resonance Lorentz model is used to describe the mean molecular polarizability. The results show that, as the propagation distance increases above a value that is on the order of an absorption depth at the pulse carrier frequency, the centroid velocity of an ultrawideband/ultrashort pulse tends toward the rate at which the Brillouin precursor travels through the medium. For small propagation distances when the carrier frequency of the optical pulse lies in the absorption band of the material, the centroid velocity can take on superluminal and negative values. © 2004 Optical Society of America

1. INTRODUCTION

Many different definitions have been introduced for the sole purpose of describing the velocity of an optical pulse in a dispersive material, the most prevalent of these being the phase, group, signal, and energy velocities. The phase velocity describes the rate at which the phase fronts of the wave propagate through the dispersive medium but can be measured only indirectly, leaving in question any separate, measurable physical meaning of this velocity measure. The group velocity, introduced by Hamilton in 1839, describes the rate at which a group of waves propagates as a whole through a dispersive medium, where a group of waves has been defined by Rayleigh as moving beats following each other in a regular pattern. With Sommerfeld’s 1914 proof that the front of a Heaviside unit step-function signal could not travel faster than the vacuum speed of light c in a Lorentz model dielectric, Brillouin defined a signal as “a short isolated succession of wavelets, with the system at rest before the signal arrived and also after it has passed” and introduced the signal velocity as the “velocity with which the main part of the wave motion propagates in the dispersive medium.” Brillouin continued, stating that “In general, the signal velocity will differ from the group velocity, especially if the phase velocity is strongly frequency dependent and if the absorption cannot be ignored.” Brillouin’s original description of the signal velocity relied upon the then newly developed asymptotic method of steepest descent and, because of a misinterpretation of the physical significance of the steepest-descent path, was in error. His result was partially corrected by Baerwald in 1930, a correct description of the signal velocity being given by Oughstun and Sherman in 1975 based upon modern asymptotic techniques that eliminate the central importance of the steepest-descent path from the analysis. Brillouin also introduced the energy velocity of a monochromatic wave, defining it as “the energy passing per second through a surface of unit area perpendicular to the direction of propagation,” which Loudon expressed as “the rate of energy flow, determined by the Poynting vector, divided by the stored energy density of the wave.” The frequency dependence of the energy velocity for a monochromatic plane wave, first derived by Loudon in 1970 for a single-resonance Lorentz model dielectric, was extended to a multiple-resonance Lorentz model dielectric by Oughstun and Shen. This relativistically correct energy velocity for a monochromatic wave has been shown to form an upper envelope to the frequency-dependent signal velocity for both a step-function signal and a rectangular envelope pulse. Although this energy transport velocity has been derived only for a monochromatic signal, it has been shown that it can be used in conjunction with the frequency dependence of the material absorption to provide a complete physical description of dispersive pulse dynamics. This new energy velocity description of dispersive pulse dynamics reduces to the group-velocity description in the low-loss limit.

Although these different velocity measures provide comparable results in those frequency regions of the material dispersion where the loss is small, they disagree wherever the material loss is large. In fact, some definitions of the pulse velocity (e.g., the phase and group velocities) yield seemingly nonphysical results (superluminal or negative velocities), while others (e.g., the group velocity) apply only to certain pulse characteristics. The phase velocity also yields superluminal values in the normal-dispersion region above the largest medium resonance frequency, so that its nonphysical high-frequency behavior is at variance with the other velocity measures discussed here. In 1970 Smith introduced the definition of the pulse centroid velocity in the hope of introducing a measurable pulse velocity that would overcome these shortcomings. This pulse centroid velocity is defined by the quantity

\[ S \left( \left. \frac{\int_{-\infty}^{\infty} t E^2(r, t) \, dt}{\int_{-\infty}^{\infty} E^2(r, t) \, dt} \right|^{-1} \right) , \] (1)

where \( E(r, t) \) is the real electric intensity vector. Note that this ratio is analogous to a center-of-mass calcula-
tion, since it tracks the temporal center of gravity of the intensity of the pulse. We consider a variant of Smith’s centroid velocity as presented by Lisak\(^2\) and further studied by Peatross et al.,\(^23,24\) which tracks the temporal center of gravity of the real Poynting vector (in mks units)

\[
\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)
\]

(2)
rather than that of the pulse intensity. Note that this is a “time-of-flight” velocity measure, which may be convenient for experimental measurement. Although Lisak refers to this velocity definition as the temporal energy velocity, in order to avoid confusion with the aforementioned energy velocity considered first by Brillouin\(^8\) and later by Loudon,\(^13\) we will refer to it here as the pulse centroid velocity of the Poynting vector or, for the sake of brevity, the centroid velocity.

For the results presented in this paper, a numerical code utilizing the fast Fourier transform (FFT) algorithm is used to determine the propagated electromagnetic pulse by means of a straightforward spectral representation. The centroid velocity of a rectangular-modulated plane-wave pulse and a Gaussian modulated plane-wave pulse, both linearly polarized and traveling in the positive \(z\) direction through a dielectric material that occupies the positive half-space \(z > 0\), is then calculated. The index of refraction of the material is represented by the Lorentz–Lorenz formula\(^4\) in which a single-resonance Lorentz model\(^1\) is used to describe the mean polarizability. These numerical results are discussed in terms of the established asymptotic theory of dispersive pulse propagation.\(^15,17,18,25-27\) Although the asymptotic theory is defined in the sense of Poincaré\(^28\) as \(z \to \infty\), the asymptotic description that results has been shown\(^29\) to provide an accurate description of the pulse dynamics in the mature dispersion regime when (typically) \(z \gg z_d\), where \(z_d\) denotes the \(\exp(-1)\) absorption depth at some appropriate characteristic frequency of the input pulse. For each pulse type considered, the description naturally separates into three frequency regions dependent upon whether the carrier frequency of the initial pulse lies below, within, or above the region of anomalous dispersion of the dispersive dielectric.

2. FORMULATION

Assume that the negative half-space \(z < 0\) is vacuum while the positive half-space \(z > 0\) is a dispersive, attenuative dielectric material, the plane \(z = 0\) being the interface between these two media. The dielectric material is nonmagnetic with magnetic permeability \(\mu = \mu_0\). The mean molecular polarizability \(\alpha_m(\omega)\) of this dielectric material is described by a single-resonance Lorentz model\(^4\) as

\[
\alpha_m(\omega) = \frac{-q_e^2/m_e}{\omega^2 - \omega_0^2 + 2i\delta\omega},
\]

(3)
where \(q_e\) is the magnitude of the electronic charge, \(m_e\) is the mass of the electron, and \(\omega_0\) is the undamped resonance frequency and \(\delta\) is the phenomenological damping constant of the dispersive material. The Lorentz–Lorenz formula\(^4\) expresses the complex index of refraction as (in mks units)

\[
n(\omega) = \left[1 + \frac{(2/3\omega_0^2)\alpha_m(\omega)}{1 - (1/3\omega_0^2)\alpha_m(\omega)}\right]^{1/2},
\]

(4)
where \(N\) is the number of harmonically bound electrons per unit volume associated plasma frequency \(b = (Nq_e^2/m_e\omega_0)^{1/2}\). For brevity, we shall sometimes refer to the model of the dielectric material as described above as the Lorenz–Lorentz model of the dielectric. The material parameters used here are \(b = \sqrt{20} \times 10^{16}\) rad/s, \(\delta = 0.28 \times 10^{16}\) rad/s, and \(\omega_0 = (16.0 \times 10^{32} + b^2/3)^{1/2} \approx 4.761 \times 10^{16}\) rad/s. Because of the approximate equivalence relation\(^30\) between the complex index of refraction described by the Lorentz–Lorenz formula\(^4\) with Eq. (3) and the complex index of refraction that results from the approximate expression \(n(\omega) \approx [1 + (4\pi/3)N\alpha(\omega)]^{1/2}\) with Eq. (3) that is used in the Lorentz model alone (but is valid only when \(N\) is sufficiently small), this choice of material parameters corresponds almost identically to the highly absorptive material described by the Lorentz model alone that is used in Brillouin’s\(^7,8\) classical analysis. The frequency dependence of the complex index of refraction for this material is illustrated in Fig. 1. The region of anomalous dispersion then begins at \(\omega_{\min} = (\omega_0^2 - b^2/3 - \delta^2)^{1/2}\) and extends to \(\omega_{\max} = (\omega_0^2 + b^2/3 - \delta^2)^{1/2}\).

We consider a field source located in a bounded region of the negative half-space that generates the two types of transverse plane-wave pulses treated in this paper, a rectangular-modulated sine wave of ten oscillations at a fixed carrier frequency and a Gaussian-modulated cosine wave of one oscillation at a fixed carrier frequency. Without loss of generality, each plane-wave field is assumed to be linearly polarized with the electric field in the \(x\) direc-

\[\text{Fig. 1. Real and imaginary parts of the complex index of refraction}\]

\[\text{for the Lorentz–Lorenz modified single-resonance Lorentz model dielectric with plasma frequency } b = \sqrt{20} \times 10^{16}\text{ rad/s, phenomenological damping constant } \delta = 0.28 \times 10^{16}\text{ rad/s, and undamped resonance frequency } \omega_0 = (16 \times 10^{32} + b^2/3)^{1/2} \approx 4.761 \times 10^{16}\text{ rad/s. Note that the Lorentz–}

\[\text{Lorenz formula shifts this undamped resonance frequency to the effective resonance frequency } \omega_0^\text{eff} = 4 \times 10^{16}\text{ rad/s.}\]
tion and the magnetic field in the y direction. Note that both of these pulses are ultrawideband; the rectangular pulse in the sense that its spectrum falls off as $\omega^{-1}$ as $\omega \to \infty$, and the Gaussian pulse in the sense that its spectral bandwidth [as measured at the exp(-1) points] spans the frequency domain from $\omega = 0$ to $\omega = 1 \times 10^{17} \text{rad/s}$, which encompasses the material resonance frequency. The Gaussian envelope pulse is also ultrashort with initial duration $T < 1 \text{fs}$, measured at the exp(-1) amplitude points, for all cases considered.

In each case considered, the pulse travels in the positive $z$ direction and is normally incident upon the dielectric interface at the plane $z = 0$. Each transmitted pulse is calculated in the spectral domain by use of the normal-incidence Fresnel transmission coefficients $^4$

$$\tau_E = \frac{\vec{E}_t}{\vec{E}_i} = \frac{2n_1(\omega)}{n_1(\omega) + n_2(\omega)},$$

$$\tau_B = \frac{\vec{B}_t}{\vec{B}_i} = \frac{2n_2(\omega)}{n_1(\omega) + n_2(\omega)},$$

where $n_1(\omega) = 1$ for the case of vacuum in the negative half-space $z < 0$, $n_2(\omega)$ is given in Eq. (4), $\vec{E}_i$ and $\vec{E}_t$ are the transmitted and incident electric field spectra, respectively, and $\vec{B}_i$ and $\vec{B}_t$ are the transmitted and incident magnetic field spectra, respectively. The frequency dependence of these two transmission coefficients for the vacuum/Lorenz–Lorentz medium interface is illustrated in Fig. 2 with our choice of the material parameters. This transmitted pulse is then taken as the initial pulse in the dispersive material for the centroid velocity calculations. Note that the transmitted field at $z = 0^+$ due to the incident rectangular envelope pulse exhibits an abrupt phase discontinuity at its trailing edge. This is due to the instantaneous change in the incident field amplitude and will not be present if the input pulse envelope has a continuous slope, as is the case for the Gaussian envelope pulse.

For a plane-wave pulse traveling in the positive $z$ direction with initial value specified at $z = 0$, the average pulse centroid velocity, which is denoted here by $v_c$, is determined by the equation

$$v_c = \frac{z}{\langle t_z \rangle - \langle t_0 \rangle},$$

where

$$\langle t_z \rangle = \frac{\int_{-\infty}^{\infty} t S(z, t) dt}{\int_{-\infty}^{\infty} S(z, t) dt}$$

is the arrival time of the temporal centroid of the Poynting vector at the plane $z = 0$. A calculation of the centroid velocity then requires knowledge of the propagated Poynting vector $S(z, t)$, which, in turn, requires expressions for $E(z, t)$ and $B(z, t) = \mu_0 H(z, t)$. Unfortunately, there are no closed-form solutions for the propagated field vectors $E(z, t)$ and $B(z, t)$ for these canonical problems (except, of course, for the trivial case of vacuum). There are, however, asymptotic approximations $^13$ that are valid throughout the mature-dispersion regime, whose accuracy increases in the sense of Poincare$^2$ as $z \to \infty$. In this paper, the behavior of the field vectors as a function of time at a set of fixed propagation distances $z$ is determined by a numerical code that is described in Section 3. The results are compared with those given by the asymptotic approximations as $z$ increases above $z_d$, where $z_d$ is the exp(-1) absorption depth of the material at the input pulse carrier frequency.

As shown by Peatross $et al.$ $^30,32$ the difference between the propagated and initial temporal centers of gravity of the Poynting vector can be expressed as the sum of two terms:

$$\langle t_z \rangle - \langle t_0 \rangle = G_z + R_0,$$

where $G_z$, called the net group delay, is given by the expression

$$G_z = \int_{-\infty}^{\infty} \frac{\partial}{\partial \omega} \frac{\partial R(z, \omega)}{\partial \omega} S(z, \omega) d\omega$$

$$= \int_{-\infty}^{\infty} S(z, \omega) d\omega,$$

and $R_0$, termed the reshaping delay, is given by

$$R_0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial \omega} \left[ \exp(3|\mathbf{K}|z) E(0, \omega) \right] \times \exp(3|\mathbf{K}|z) H^*(0, \omega) d\omega$$

$$= i \int_{-\infty}^{\infty} \exp(23|\mathbf{K}|z) S(0, \omega) d\omega.$$
The numerical method used to calculate the Poynting vector at any plane \( z > 0 \) utilizes the fast Fourier transform (FFT) algorithm to determine the propagated electric and magnetic fields. To numerically determine the linearly polarized propagated electric field vector \( \mathbf{E}(z, t) = \mathbf{\hat{x}} \mathbf{E}(z, t) \), this code uses the FFT to compute the spectrum \( \mathbf{E}(0, \omega) \) of the initial electric field vector \( \mathbf{E}(0, t) = \mathbf{\hat{x}} \mathbf{E}(0, t) \), which is then multiplied by the Fresnel transmission coefficient \( \tau_E(\omega) \). Each transmitted monochromatic component is then propagated by multiplication with the propagation factor \( \exp(i \mathbf{\hat{k}}(\omega)z) \), where \( \mathbf{\hat{k}}(\omega) = (\omega/c)n(\omega) \) is the complex wave number in the medium with complex index of refraction \( n(\omega) \) at the angular frequency \( \omega \). The temporal structure of the propagated electric field is then constructed by computing the inverse FFT of the propagated spectrum, so that

\[
\mathbf{E}(z, t) = \int_{-\infty}^{\infty} \tau_E(\omega) \mathbf{E}(0, \omega) \exp[i(\mathbf{\hat{k}}(\omega)z - \omega t)] d\omega. \tag{12}
\]

Likewise, the propagated magnetic field vector \( \mathbf{B}(z, t) = \mathbf{\hat{y}} \mathbf{B}(z, t) \) is numerically determined by multiplying the initial spectrum \( \mathbf{B}(0, \omega) \) by \( (1/c)n(\omega)\tau_B(\omega) \). Each transmitted monochromatic component is propagated by using the same propagation factor \( \exp(i \mathbf{\hat{k}}(\omega)z) \), and the temporal structure of the propagated magnetic field is then constructed by computing the inverse FFT of the propagated spectrum, so that

\[
\mathbf{B}(z, t) = \frac{1}{c} \int_{-\infty}^{\infty} n(\omega) \tau_B(\omega) \mathbf{E}(0, \omega) \exp[i(\mathbf{\hat{k}}(\omega)z - \omega t)] d\omega, \tag{13}
\]

for a given fixed value of \( z > 0 \). If the initial pulse is given within the dispersive material, then \( \tau_E \) and \( \tau_B \) are replaced by unity in Eqs. (12) and (13). Note that, because of the frequency dispersion of \( n(\omega) \), and hence of \( \tau_E(\omega) \), if the input pulse frequency is unchirped, then the transmitted pulse will, to some extent, become chirped, the amount dependent upon the value of the input pulse frequency relative to the material resonance frequency.

The accuracy of this numerical approach depends directly upon the highest frequency necessary to accurately describe the initial pulse spectrum, and the highest frequency necessary to accurately describe the material dispersion. The maximum frequency sampled in our calculations is at least \( 2 \pi \times 10^{12} \) rad/s with at least \( 2^{17} \) points sampled; a higher maximum frequency with more sample points is used when the applied signal frequency is set above resonance. The initial pulse spectrum and the material dispersion are accurately modeled in all cases considered.

4. ASYMPTOTIC DESCRIPTION

In this paper, the initial electric field pulse \( E(0, t) \) is described by a pulse-modulated sine or cosine wave of fixed applied frequency \( \omega_0 \):

\[
E(0, t) = u(t) \sin(\omega_0 t + \psi), \tag{14}
\]

where \( u(t) \) is the real-valued initial envelope function with either \( \psi = 0 \) for a sine-wave carrier or \( \psi = \pi/2 \) for a cosine-wave carrier. The integral representation of the propagated plane-wave pulse can then be expressed in terms of the scalar wave field \( A(z, t) \) given by

\[
A(z, t) = \frac{1}{2\pi} \Re \left\{ i \exp(-i\psi) \right. \times \int_{ia}^{ia+\infty} \tau_E(\omega) \tilde{u}(\omega - \omega_c) \exp\left[ \frac{z}{c} \phi(\omega, \theta) \right] d\omega \right\} \tag{15}
\]

for \( z > 0 \), where \( \mathbf{E}(z, t) = \mathbf{\hat{x}} A(z, t) \) and \( \tilde{u}(\omega) \) is the temporal Fourier spectrum of the initial pulse envelope function \( u(t) \). The spatial evolution of the Fourier spectrum \( \tilde{A}(z, \omega) \) of the temporal pulse \( A(z, t) \) satisfies the dispersive Helmholtz equation

\[
[\nabla^2 + k^2(\omega)] \tilde{A}(z, \omega) = 0, \tag{16}
\]

with complex wave number \( k(\omega) = (\omega/c)n(\omega) \), where \( c \) is the speed of light in vacuum. The quantity \( a \) appearing in Eq. (15) is a real number greater than the abscissa of absolute convergence\(^\dagger\) for the initial pulse function \( A(0, t) = E(0, t) \). The complex phase function \( \phi(\omega, \theta) \) appearing in Eq. (15) is defined as

\[
\phi(\omega, \theta) = \frac{c}{z} \left[ k(\omega)z - \omega t \right] = i \omega [n(\omega) - \theta] \tag{17}
\]

where \( \theta = ct/z \) is a dimensionless space–time parameter. This formulation is identical with that given in Eqs. (12) and (13) when \( E(0, t) \) is given by Eq. (14). The asymptotic description\(^7,8,15,18\) is obtained as a function of the dimensionless space–time parameter \( \theta \) as \( z \to \infty \). For a sufficiently large fixed value of \( z \), it yields the asymptotic description of the temporal pulse evolution at that fixed propagation distance.

If \( u(t) = 0 \) for \( t < 0 \), then this integral can be analytically determined for all \( \theta < 1 \) by completing the contour in the upper half of the complex \( \omega \) plane and applying Cauchy’s theorem\(^2\) to yield\(^5,6,15,18\)

\[
A(z, t) = 0. \tag{18}
\]
ever, no such evaluation is possible for values of \( \theta \gg 1 \), and asymptotic methods are then required to obtain an analytic evaluation of Eq. (15).

As described by Oughstun and Sherman,\textsuperscript{15,17,18,25–27} the asymptotic expansion of \( A(z, t) \) for \( \theta \gg 1 \) in a single-resonance Lorentz model dielectric consists either of three contributions to the integral, given by

\[
A(z, t) = A_S(z, t) + A_B(z, t) + A_C(z, t), \quad (18)
\]

or of a superposition of fields of this form. Here \( A_S(z, t) \), \( A_B(z, t) \), and \( A_C(z, t) \) are referred to as the Sommerfeld precursor, the Brillouin precursor, and the main signal, respectively. These asymptotic results\textsuperscript{15,18,26} show that the magnitude of both the Sommerfeld precursor and the main signal decay exponentially, while the magnitude of the peak of the Brillouin precursor decays only algebraically as \( z^{-1/2} \) as \( z \to \infty \). Because of the nonexponential, algebraic decay of the peak amplitude in the Brillouin precursor and the exponential decay of the other factors, the Brillouin precursor will dominate the other contributions to the field for sufficiently large propagation distances.\textsuperscript{33} Since the peak of the Brillouin precursor occurs at the space–time point \( \theta = n(0) = \theta_0 \), this peak amplitude point then travels through the dispersive material at the rate \( c/\theta_0 = c/n(0) \). For the material parameters chosen here, \( \theta_0 = 3/2 \) and \( c/\theta_0 = (2/3)c \).

With the determination of the asymptotic behavior of \( A(z, t) \) as \( z \to \infty \), the asymptotic behavior of the electric field vector \( \mathbf{E}(z, t) \) as \( z \to \infty \) is directly determined from the relation \( \mathbf{E}(z, t) = \mathbf{S}A(z, t) \) for a linearly polarized plane wave traveling in the positive \( z \) direction. To determine the asymptotic behavior of the magnetic field vector \( \mathbf{B}(z, t) \) as \( z \to \infty \), the same technique is applied with the multivalued complex index of refraction \( n(\omega) \) introduced into the integrand of Eq. (15) through the factor \((1/c)n(\omega)\) [cf. Eq. (13)]. This extra factor introduces no additional complications into the analysis, since the location of the poles of \( n(\omega) \) are the same as the branch-cut locations for \( \phi(\omega, \theta) \), which are naturally avoided in the asymptotic analysis.\textsuperscript{15,18} With the asymptotic approximations for both \( \mathbf{E}(z, t) \) and \( \mathbf{B}(z, t) \) known, the Poynting vector is then directly calculated as

\[
\mathbf{S}(z, t) = (1/\mu_0)\mathbf{E}(z, t)\mathbf{B}(z, t)\hat{z} \quad (19)
\]

for plane-wave pulse propagation in the positive \( z \) direction. Since the asymptotic behavior of the magnetic field is similar to that for the associated electric field, the Poynting vector will then exhibit similar characteristics. In particular, the Brillouin precursor will dominate the structure of the Poynting vector for sufficiently large propagation distances. The pulse centroid velocity of the Poynting vector will then approach the value \( c/\theta_0 = c/n(0) \) at which the peak amplitude point of the Brillouin precursor travels through the dispersive medium as \( z \to \infty \).

5. CENTROID VELOCITY OF A RECTANGULAR MODULATED SINE WAVE

Let the initial electric field vector of a plane-wave pulse normally incident upon the dielectric interface at \( z = 0^+ \) be described by a rectangular-envelope-modulated sine wave with fixed carrier frequency \( \omega_c \). The transmitted electric and magnetic field vectors at \( z = 0^+ \), and hence the transmitted Poynting vector, will then experience a frequency chirp that is due to the frequency dependence of the material refractive index. An estimate of the average refractive-index-induced frequency chirp over each half-cycle of the main body of the pulse is found to be less than 5% of the carrier frequency \( 2\omega_c \) of the input pulse Poynting vector for all cases considered here, the greatest amount of chirp appearing at the leading edge of the pulse.

A. Asymptotic Behavior

The asymptotic description of the rectangular-modulated sine wave of initial pulse duration \( T \) is constructed from the difference of two unit step-function signals separated by the initial time duration \( T \). Thus, by defining the initial envelope function \( u(t) \) as a unit step function with \( \psi = 0 \), one can express the propagated pulse as the difference between the leading and trailing edge fields\textsuperscript{17,18}:

\[
A(z, t) = [A_S(z, t, 0) + A_B(z, t, 0) + A_C(z, t, 0)] - [A_S(z, t, T) + A_B(z, t, T) + A_C(z, t, T)]. \quad (20)
\]

The third variable, \( * \), in the triple \((z, t, *)\) appearing in this expression indicates the time at which the unit step function is turned on, where

\[
A(z, t, T) = \frac{1}{2\pi} \Im \left\{ \exp(-i\omega_c T) \right. \\
\times \int_{i\omega - \infty}^{i\omega + \infty} \tau_{\mathbf{E}}(\omega) \tilde{u}(\omega - \omega_c) \\
\times \exp\left[ \frac{z}{c} \phi_{\tau}(\omega, \theta) \right] d\omega \right\}, \quad (21)
\]

with

\[
\phi_{\tau}(\omega, \theta) = i\omega[n(\omega) - \theta_T] \\
= i\omega \left[ n(\omega) - \frac{c(T - T)}{z} \right]. \quad (22)
\]

A detailed description of the resultant pulse dynamics in a single-resonance Lorentz model dielectric may be found in Refs. 17 and 18. The propagated field structure is then described by leading- and trailing-edge precursor fields together with the signal contribution \( A_C(z, t, 0) - A_C(z, t, T) \).

B. Numerical Results

1. Carrier Frequency below the Anomalous Dispersion Regime

When the carrier frequency of the pulse lies in the normal-dispersion region below the anomalous-dispersion region of the Lorentz–Lorentz model dielectric [i.e., when \( 0 < \omega_c < (\omega_c^2 - b^2)^{1/2} - (\delta^2)^{1/2} \)], the amplitudes of both the Sommerfeld precursor and the pole contribution are negligible in comparison with the amplitude of the Brillouin precursor for propagation distances greater than a value on the order of a single absorption depth at the ap-
plied pulse frequency. \(^{17}\) Thus, from Eq. (20), the field will be dominated by the two Brillouin precursor terms \(A_B(z, t, 0)\) and \(A_B(z, t, T)\). Initially, these leading- and trailing-edge Brillouin precursors are separated in time by the initial pulse duration, so that the temporal center of gravity of the propagated Poynting vector will lie between the two peaks of these Brillouin precursors, thereby yielding a pulse centroid velocity that is slower than \(c/\theta_0 = c/n(0)\). As the propagation distance increases, the peak of the trailing-edge Brillouin precursor approaches the peak of the leading-edge Brillouin precursor \(^{17}\) [cf. Eq. (22)] and the temporal center of gravity of the propagated Poynting vector approaches the value at which the peak of the front precursor occurs, i.e., \(\langle t_z \rangle \rightarrow \theta_0 = n(0)\). Hence the asymptotic description predicts that the centroid velocity of the pulse approaches the value \(c/\theta_0 = c/n(0)\) as \(z\) increases.

The centroid velocity of the rectangular-modulated sine wave of ten oscillations at the carrier frequency \(\omega_c\) was computed with the numerical code described in Section 3 for the below-resonance cases \(\omega_c = 1 \times 10^{16}, \omega_c = 2 \times 10^{16}\), and \(\omega_c = 3 \times 10^{16} \text{ rad/s}\). For each fixed carrier frequency case, the centroid velocity was calculated for propagation distances from 0.1 to 100 absorption depths at the input pulse carrier frequency \(\omega_c\). The results are presented in Fig. 3, where the relative centroid velocity values are plotted as a function of the relative propagation distance \(z/z_d\) on a semilogarithmic scale, where \(z_d = \alpha^{-1}(\omega_c) = \exp(-1)\) penetration depth in the dispersive material at the input carrier frequency and \(\alpha(\omega) = 3k(\omega)\). The data points for the cases \(\omega_c = 1 \times 10^{16}, \omega_c = 2 \times 10^{16}\), and \(\omega_c = 3 \times 10^{16} \text{ rad/s}\) are represented by circles, asterisks, and plus signs, respectively, and the solid curves are cubic spline fits to each separate set of data points. Evident in the figure is the asymptotic approach to the limiting value \(v_c/c = 1/\theta_0 = 2/3\) as \(z \rightarrow \infty\), in agreement with the asymptotic description. In each below-resonance case considered, the centroid velocity monotonically approaches the asymptotic value \(v_c/c = 1/\theta_0\) from below as the propagation distance increases.

In the opposite limit as \(z \rightarrow 0^+\), the centroid velocity approaches the classical group velocity \(v_g(\omega_c)\), where \(v_g = (\omega/\beta)\) with \(\beta(\omega) = \Re(k(\omega))\).

When considered in terms of the net group and reshaping delays described in Eqs. (9)–(11), the net group delay is found to be dominant over the reshaping delay for all propagation distances considered. In fact, the relative error between the two velocity calculations \(v_c = z/(G_z + R_0)\) and \(v = z/G_z\) is found to be less than 2\% in all cases considered, where this relative error is monotonically decreasing with increasing propagation distance.

2. Carrier Frequency in the Region of Anomalous Dispersion

The Brillouin precursor remains the dominant contribution to the integral given in Eq. (15) for propagation distances greater than a value on the order of a single absorption depth at the applied frequency when the carrier frequency of the pulse lies within the absorption band of the material, \(^{17}\) i.e., when \((\omega_c^2 - b^2/3 - \delta^2)^{1/2} < \omega_c < (\omega_c^2 + 2b^2/3 - \delta^2)^{1/2}\). The centroid velocity for this case was calculated for the carrier frequency values \(\omega_c = 4 \times 10^{16}, \omega_c = 5 \times 10^{16}\) and \(\omega_c = 6 \times 10^{16} \text{ rad/s}\) with the results represented in Fig. 4 by circles, asterisks, and plus signs, respectively, where the solid curves are cubic spline fits to each set of data points over each of their respective domains of continuity. For propagation distances below two absorption depths, the centroid velocity for both the \(\omega_c = 4 \times 10^{16}\) and \(\omega_c = 5 \times 10^{16} \text{ rad/s}\) cases yields both negative and superluminal values before settling into an approach to the asymptotic rate of \((2/3)c\).

The extreme centroid velocity values obtained here are not indicative of pulse movement, as evident in Figs. 5 and 6. The top diagram of Fig. 5 shows the initial (dashed curve) and propagated (solid curve) Poynting vectors for the rectangular-modulated pulse with on-resonance carrier frequency \(\omega_c = 4 \times 10^{16} \text{ rad/s}\) at a propagation distance of 1.24 absorption depths. The initial pulse is taken as the pulse just transmitted across the interface within the dielectric at \(z = 0^+\). The leading-
and trailing-edge Brillouin precursors are evident in the propagated Poynting vector. The bottom diagram of Fig. 5 presents a magnification of the area that contains the calculated center of gravities. The centroid of the initial Poynting vector is denoted by the circle, while the centroid of the propagated Poynting vector is denoted by the asterisk. The computed centroid velocity at this propagation distance is $-26.2c$. Figure 6 is analogous to Fig. 5 but has been calculated at the propagation distance of 1.25 absorption depths, at which point the computed centroid velocity is $84.3c$. Comparison of the top graphs of these two figures shows that the pulse has not significantly moved between these two propagation distances. What has occurred is that the amplitude of the leading-edge precursor has decreased relative to the amplitude of the trailing-edge precursor. It is this slight rearrangement in the temporal pulse structure that results in excessive centroid velocity rates and not any superluminal motion of the pulse through the material.

If one considers the net group and reshaping delays for carrier frequencies in the absorption band of the dielectric, the net group delay is found to be of the same order as that of the reshaping delay for small propagation distances and not until propagation distances are large enough (typically $z/z_d > 10$) does the net group delay dominate the reshaping delay. Calculations of the velocity $v_c = z/G_z$ ignoring the reshaping delay can have a relative error of over 1000% when compared with the correct values given by $v_c = z/(G_z + R_0)$.

As an illustration, the net group and reshaping delays for a rectangular-modulated pulse with on-resonance applied frequency $\omega_c = 4 \times 10^{16}$ rad/s are illustrated in Fig. 7. The solid curve in this figure is a cubic spline fit to the calculated net group delays, while the dashed curve is a cubic spline fit to the reshaping delays. Apparent in the figure is the importance of both the net group and reshaping delays for small absorption depths. In general, the reshaping delay effect is dependent upon both the material parameters and the initial pulse shape.

### 3. Carrier Frequency above the Region of Anomalous Dispersion

For carrier frequencies that lie in the normal-dispersion regime $\omega_c > (\omega_0^2 + 2b^2/3 - \delta^2)^{1/2}$ above the region of anomalous dispersion of the Lorenz–Lorentz model dielectric, the Brillouin precursor contribution again dominates the integral representation of the propagated pulse given in Eq. (15) for propagation distances greater than a value on the order of a single absorption depth at the applied frequency of the pulse. However, the large amount of the input pulse spectral energy that resides in the high-frequency range results in significant leading- and trailing-edge Sommerfeld precursors and, since $n(\omega) = 1 - b^2/(2\omega^2) + i\delta b^2/\omega^3 \approx 1$ for frequencies in this

Fig. 7. Net group delay (solid curve) and reshaping delay (dashed curve) of a rectangular-envelope-modulated sine wave of ten oscillations with on-resonance carrier frequency $\omega_c = 4 \times 10^{16}$ rad/s as a function of the relative propagation distance $z/z_d$ in the Lorenz–Lorentz model dielectric.
range, these Sommerfeld precursors remain significant in all of the cases considered here. The front of the Sommerfeld precursor travels at the vacuum speed of light \( c \) and quickly ascends to its peak amplitude soon after that point, while the peak of the Brillouin precursor occurs at \( \theta_0 = 3/2 \). As a consequence, the asymptotic behavior of the temporal center of gravity of the propagated Poynting vector starts at a space–time value of \( \theta = ct/z \) that is above the value \( \theta = \theta_0 = 3/2 \) and slowly decreases to this limit as the propagation distance increases and the leading- and trailing-edge Sommerfeld precursors gradually decay in amplitude.

The centroid velocity was calculated for the two carrier frequency cases \( \omega_c = 8 \times 10^{16} \) and \( \omega_c = 1 \times 10^{17} \) rad/s, both of which lie above the region of anomalous dispersion. The resultant relative centroid velocities are illustrated in Fig. 8 for propagation distances from 0.1 to 100 absorption depths for each applied carrier frequency case. The figure clearly shows that the centroid velocity begins at the classical group velocity value \( v_c(\omega) \) when \( z = 0 \), rapidly increases to a subluminal peak value, and then descends to the limiting value \( \lim_{z \to \infty} (v_c)/c = 2/3 \), in agreement with the asymptotic theory.

For applied carrier frequencies above the absorption band of the material, the net group delay is again dominant over the reshaping delay. When the velocity of the pulse is recalculated without the reshaping delay, the relative error is found to be less than 5\% in all cases considered, where this error monotonically decreases with increasing propagation distance.

6. CENTROID VELOCITY OF AN ULTRASHORT GAUSSIAN-ENVELOPE-MODULATED COSINE WAVE

Let the initial electric field vector of a plane-wave pulse normally incident upon the dielectric interface at \( z = 0 \) be described by a single-cycle Gaussian-envelope-modulated cosine wave with fixed carrier frequency \( \omega_c \). The transmitted electric and magnetic field vectors at \( z = 0^+ \), and hence the transmitted Poynting vector, will then experience a frequency chirp due to the frequency dependence of the material refractive index. An estimate of the average refractive-index-induced frequency chirp of the transmitted Poynting vector is found to be less than 5\% of the input Poynting vector carrier frequency \( 2\omega_c \) at the center of the single-oscillation pulse for all cases considered.

A. Asymptotic Description

The asymptotic expansion of the Gaussian-modulated cosine wave is now considered. Here, \( \psi = \pi/2 \) [see Eq. (14)], and, for an initial pulse centered at \( t_0 \), the initial pulse envelope is given by

\[
u(t) = \exp\left[-\frac{(t - t_0)^2}{T^2}\right]
\]

with initial envelope spectrum

\[
\tilde{u}(\omega) = \pi^{1/2}T \exp(i\omega t_0) \exp\left(-\frac{T^2\omega^2}{4}\right),
\]

where \( 2T \) is the initial pulse width. Because \( \tilde{u}(\omega) \) is an entire function of \( \omega \), there is no pole contribution to Eq. (15) and the asymptotic expansion of \( A(z, t) \) consists of just the Sommerfeld and Brillouin precursor fields,\(^{34-36} \) so that

\[
A(z, t) = A_S(z, t) + A_B(z, t).
\]

Explicit analytic expressions for these so-called “Gaussian” precursor fields may be found in Refs. 34 and 36.

B. Numerical Results

1. Carrier Frequency below the Region of Anomalous Dispersion

When the carrier frequency of the pulse lies in the normal-dispersion region below the region of anomalous dispersion of the Lorenz–Lorentz model dielectric, the amplitude of the Sommerfeld precursor is negligible compared with that of the Brillouin precursor for propagation distances greater than a value on the order of a single absorption depth at the applied frequency of the pulse. As a consequence, the centroid velocity will rapidly approach the asymptotic limit \( v_c = c/\theta_0 \) as \( z \to \infty \), which is the rate at which the Brillouin precursor travels through the material.

The centroid velocity of a single-oscillation Gaussian-modulated pulse was numerically determined for the below-resonance carrier frequency cases \( \omega_c = 1 \times 10^{16} \), \( \omega_c = 2 \times 10^{16} \), and \( \omega_c = 3 \times 10^{16} \) rad/s over propagation distances from 0.1 to 100 absorption depths using the numerical propagation code described in Section 3. The results are summarized in Fig. 9, where circles, asterisks, and plus signs denote the data points for \( \omega_c = 1 \times 10^{16} \), \( \omega_c = 2 \times 10^{16} \), and \( \omega_c = 3 \times 10^{16} \) rad/s, respectively, and the solid curves are cubic spline fits to these data points. The limit \( v_c/c = 1/\theta_0 = 1/(n(0)) = 2/3 \) as \( z \to \infty \) is clearly evident in this figure, in agreement with the asymptotic theory. In addition, the centroid velocity is seen to approach the classical group velocity value \( v_g(\omega_c) \) as \( z \to 0^+ \). In terms of the net group and reshaping delays, the net group delay begins and remains dominant.
The numerical results are presented in Fig. 10, where circles, asterisks, and plus signs represent the data points for \( \omega_c = 4 \times 10^{16} \), \( \omega_c = 5 \times 10^{16} \), and \( \omega_c = 6 \times 10^{16} \) rad/s, respectively, and the solid curves are cubic spline fits to each set of data points. As evident in the figure, the centroid velocity rapidly approaches the asymptotic limit \( v_c = c/\theta_0 = (2/3)c \) as \( z \to \infty \). Comparison of these results with those presented in Fig. 4 for the corresponding rectangular envelope pulse cases shows that the asymptotic behavior is obtained at smaller propagation distances for the rectangular envelope pulses than it is for the corresponding Gaussian envelope pulse cases.

Again, as for the rectangular pulse, the net group delay is of the same order of magnitude as that of the reshaping delay for small propagation distances when the carrier frequency of the pulse lies within the absorption band of the material. If the velocity is calculated by using only the net group delay \( v_c = z/Gz \), there may be more than a 20% relative error out to five absorption depths at the pulse carrier frequency, and this error monotonically decreases with increasing propagation distance.

3. Carrier Frequency above the Region of Anomalous Dispersion

For carrier frequencies that lie in the normal-dispersion region above the absorption band of the material, both the Sommerfeld and Brillouin precursors are evident during the pulse propagation for propagation distances greater than a value on the order of a single absorption depth at the applied frequency of the pulse. Again, the large amount of spectral energy situated in the high-frequency domain means that the Sommerfeld precursor will be a significant component of the propagated pulse, even for large propagation distances. The peak of the Sommerfeld precursor travels at a velocity just below \( c \), while the peak of the Brillouin precursor travels at the velocity \( \theta_0c = (2/3)c \). Thus the value of the pulse centroid velocity will start above \( (2/3)c \) and slowly descend to this
asymptotic limit as the Sommerfeld precursor gradually decays in amplitude with increasing propagation distance.

The relative centroid velocity values for the Gaussian-modulated pulse cases with carrier frequencies \( \omega_c = 8 \times 10^{16} \) and \( \omega_c = 1 \times 10^{17} \) rad/s, which correspond to frequencies above the region of anomalous dispersion, are presented in Fig. 11 for propagation distances that range from 0.1 to 100 absorption depths at the applied carrier frequency. The graph clearly shows in each case considered that the centroid velocity starts above and then descends to the limit of \((v_c)/c = 2/3\), in agreement with the asymptotic theory. Comparison of these results with those presented in Fig. 8 for the corresponding rectangular-envelope pulse cases shows that the asymptotic behavior is observed at smaller propagation distances for the Gaussian envelope pulses than it is for the rectangular envelope cases. This reversal from the behavior observed for both the below-resonance and on-resonance cases is simply due to the fact that the temporal width \(2T\) of the input single-cycle Gaussian envelope pulse decreases as the carrier frequency \(\omega_c\) increases, where \(2T = 2\pi/\omega_c\).

Here, again, the net group delay is dominant over the reshaping delay for the Gaussian-modulated pulse for all cases considered. The relative error between calculating the centroid velocity with or without the reshaping delay is less than 14%, and this error decreases with increasing propagation distance.

7. INSTANTANEOUS CENTROID VELOCITY

The above numerical results are for an average centroid velocity of the Poynting vector that is determined by the initial and final centroid locations within the dielectric material. A natural question to ask is whether or not the instantaneous centroid velocity would exhibit the same behavior as that of the average centroid velocity. The instantaneous centroid velocity of the Poynting vector is defined here as

\[
v_{ic} = \lim_{z_2 \to \infty} \frac{z_2 - z_1}{z_2 - z_1},
\]

where \((t_j)\) is the centroid of the Poynting vector at the propagation distance \(z_j\). A numerical estimate of this expression is obtained by selecting neighboring points \((z_1, z_2)\) in the above centroid velocity data sets that are sufficiently close to yield an accurate estimate of the limiting expression appearing in Eq. (26).

The instantaneous centroid velocity was calculated for both the rectangular- and Gaussian-modulated pulses for each of the cases considered for the average centroid velocity. With the exception of the rectangular-modulated pulse with below-resonance carrier frequency, the instantaneous centroid velocity results are qualitatively similar to the average centroid velocity results. In particular, as the propagation distance increases, the limiting value \(v_{ic}/c \to 1/\theta_0 = 0.67\) is always obtained. For the rectangular-modulated pulse cases with below-resonance carrier frequencies, the instantaneous centroid velocity now peaks to a maximum value between \(z/z_d = 2\) and \(z/z_d = 3\) and then approaches the limit \(v_{ic}/c \to 1/\theta_0 = 0.67\) from above, as seen in Fig. 12 (compare with Fig. 3). This peak value increases as the initial pulse carrier frequency increases through the below-resonance normal-dispersion region and just becomes superluminal about the carrier frequency value \(\omega_c = 0.75\omega_0\). The results presented in Fig. 12 show that the pulse initially "accelerates" until its instantaneous centroid velocity reaches a peak value between \(z/z_d = 2\) and \(z/z_d = 3\), after which it "decelerates" toward the asymptotic value \(c/\theta_0\) set by the velocity of the peak amplitude point of the Brillouin precursor.

8. CONCLUSIONS

The evolution of the pulse centroid velocity of the Poynting vector for both ultrawideband rectangular-envelope-modulated and ultrashort Gaussian-envelope-modulated plane-wave pulses traveling through a Lorentz–Lorenz modified Lorentz model dielectric with a single-resonance frequency has been presented. Each initially unchirped pulse is normally incident from vacuum upon the plane interface of the dispersive material. The frequency-dependent transmission coefficient introduces a frequency chirp into the transmitted pulse, which then influences the observed initial centroid velocity evolution. The results lead to the following conclusions:

1. As \(z \to \infty\), \(v_c \to c/\theta_0 = c/n(0)\), and the average pulse centroid velocity of an ultrawideband/ultrashort pulse tends toward the rate at which the peak of the Brillouin precursor travels through the medium, independent of the initial pulse carrier frequency, in agreement with the asymptotic theory. The same result holds for the instantaneous pulse centroid velocity.

2. As \(z \to 0\), the average pulse centroid velocity \(v_c\) approaches the classical group velocity \(v_{g}(\omega_c)\) at the input pulse carrier frequency when that frequency is in either of the normal-dispersion regions above or below the material absorption band where \(v_{g}/c < 1\). The same result holds for the instantaneous pulse centroid velocity.
3. Although the average pulse centroid velocity of an ultrashort Gaussian envelope pulse remains subluminal for propagation distances above one tenth of an absorption depth (a result comforting to the authors), the trend to superluminal group velocity values in the limit as \( z \to 0 \) when the input pulse carrier frequency is in the region of anomalous dispersion was not obtained because the required numerical accuracy exceeded that provided by our computer.

4. Superluminal average pulse centroid velocity values are observed for input rectangular envelope pulses when the carrier frequency is within the absorption band (region of anomalous dispersion) of the material for propagation distances on the order of an absorption depth \( (z/z_d \sim 1) \). The trend to superluminal group velocity values in the limit as \( z \to 0 \) in the anomalous-dispersion region was not obtained because the required numerical accuracy exceeded that of our computer.

5. Superluminal instantaneous pulse centroid velocity values are obtained for the rectangular envelope pulse with below-resonance carrier frequency values about \( \omega_c \sim 0.75\omega_0 \) and for carrier frequency values within the absorption band of the material.

6. In certain cases, the centroid velocity of the Poynting vector may not accurately describe the pulse velocity through the material with regard to energy transport. For example, when the Sommerfeld precursor amplitude is of the same order as that of the Brillouin precursor amplitude, the centroid velocity will fall between the two precursors at a point where a negligible amount of pulse energy may be located.

7. The reshaping delay is significant [i.e., produces a relative error of over 20\% between the two centroid velocity calculations \( v_c = z/G_z \) and \( v_c = z/(G_z + R_\theta) \)] for small propagation distances when the carrier frequency of the pulse lies within the region of anomalous dispersion for the material parameters used here. For a rectangular-modulated pulse with carrier frequency within the absorption band of the material, the reshaping delay was found to be significant for propagation distances less than or equal to ten absorption depths at the carrier frequency of the pulse. For the Gaussian-modulated pulse with carrier frequency within the absorption band of the material, the reshaping delay was found to be significant for propagation distances less than or equal to five absorption depths at the carrier frequency of the pulse. In general, the relative significance of the reshaping delay is dependent upon both the material parameters and the initial pulse envelope.

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N. A. Cartwright’s e-mail address is ncartwri@emba.uvm.edu.

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33. Here, “dominates the field” refers to the amplitude of the Brillouin precursor being larger than the other contributions to the field when the entire propagated field is considered, as it is in calculating the centroid velocity. However, there are θ domains within the evolved pulse in which the Sommerfeld precursor or the pole contribution is the dominant contribution to the field.

