Polarization properties of the freely propagating electromagnetic field of arbitrary spatial and temporal form

Kurt Edmund Oughstun

Department of Computer Science and Electrical Engineering and Department of Mathematics and Statistics, University of Vermont, Burlington, Vermont 05405

Received January 30, 1991; revised manuscript received November 1, 1991; accepted November 6, 1991

The angular spectrum of plane waves representation of the freely propagating electromagnetic field in the half-space \( z \geq z_0 \) of a homogeneous, isotropic, locally linear, temporally dispersive medium is employed to describe the polarization properties of the field throughout the half-space \( z > z_0 \) in terms of the polarization behavior of either field vector on the plane at \( z = z_0 \). Both the spatial and the temporal dependence of the initial field vectors on this plane are assumed to be known but are otherwise arbitrary functions of the transverse position vector \( \mathbf{r}_T \) and time \( t \) on that plane. The general relationship between the electric and the magnetic polarization ellipses is obtained, along with the general orthogonality relations between the field vectors. Finally, the conditions under which either or both field vectors may be uniformly polarized are described.

1. INTRODUCTION

The usual treatment of the polarization properties of the electromagnetic field\(^1\) is restricted to the special case of a time-harmonic plane-wave field. This restriction has, in part, been removed by Nisbet and Wolf\(^6\) for the case of linearly polarized, time-harmonic waves of arbitrary form and, more recently, by Fainman and Shamir\(^8\) in an analysis of the effects of polarizers on nonplanar wave fronts. The analysis of the properties of a general, freely propagating electromagnetic field is presented in this paper by using the angular spectrum of plane waves representation.\(7,12\) The term freely propagating is used here to indicate that there are no externally supplied charge or current sources for the field in the region of space under consideration. This should not be confused with a free field whose externally supplied sources have all been turned off\(7,8\) (a general description of the angular spectrum of plane waves representation for both radiation and free fields may be found in this referenced body of work) or with a source-free field whose plane-wave expansion contains only homogeneous waves.\(11\) The type of field considered here is more general than that of a source-free field in that it may contain both homogeneous and inhomogeneous plane-wave spectral components. In addition, the analysis of the present paper allows for the more general situation in which the electromagnetic field is pulsed in time and is propagating in a general temporally dispersive and absorptive dielectric medium with a frequency-dependent conductivity. A related analysis for the time-harmonic case has been presented by Borgiotti\(^9\) for obtaining a given behavior of polarization in the radiation pattern that is due to an aperture antenna.

Consider an electromagnetic field that is propagating into the source-free half-space \( z \geq z_0 > 0 \) of a homogeneous, isotropic, locally linear, temporally dispersive medium with a (complex-valued) frequency-dependent dielectric permittivity \( \varepsilon(\omega) \) and electric conductivity \( \sigma(\omega) \) and a (real-valued) constant magnetic permeability \( \mu \).

Let the field vectors on the plane \( z = z_0 \), denoted by

\[ \mathbf{A}(\mathbf{r}_T, z_0, t) = \mathbf{A}_0(\mathbf{r}_T, t), \]

\[ \mathbf{B}(\mathbf{r}_T, z_0, t) = \mathbf{B}_0(\mathbf{r}_T, t), \]

be known functions of time and the transverse position vector \( \mathbf{r}_T = \hat{\mathbf{i}}_x x + \hat{\mathbf{i}}_y y \) in the plane at \( z = z_0 \), as illustrated in Fig. 1. The source of this field resides somewhere in the region \( z < z_0 \). The angular spectrum representation of the freely propagating electromagnetic field vectors throughout the half-space \( z \geq z_0 \) is then given by\(12\)

\[ \mathbf{A}(\mathbf{r}_T, t) = \frac{1}{4\pi^2} \Re \int_{C_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{E}_0(\mathbf{k}_T, \omega) \times \exp[i(\mathbf{k}_T \cdot \mathbf{r}_T - \omega t)] d\mathbf{k}_x d\mathbf{k}_y, \]

\[ \mathbf{B}(\mathbf{r}_T, t) = \frac{1}{4\pi^2} \Re \int_{C_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{B}_0(\mathbf{k}_T, \omega) \times \exp[i(\mathbf{k}_T \cdot \mathbf{r}_T - \omega t)] d\mathbf{k}_x d\mathbf{k}_y, \]

with \( \mathbf{r}_T = \hat{\mathbf{i}}_T z + \hat{\mathbf{i}}_y y \), where

\[ \mathbf{E}_0(\mathbf{k}_T, \omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \mathbf{E}_0(\mathbf{r}_T, t) \times \exp[-i(\mathbf{k}_T \cdot \mathbf{r}_T - \omega t)] d\mathbf{x} d\mathbf{y}, \]

\[ \mathbf{B}_0(\mathbf{k}_T, \omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \mathbf{B}_0(\mathbf{r}_T, t) \times \exp[-i(\mathbf{k}_T \cdot \mathbf{r}_T - \omega t)] d\mathbf{x} d\mathbf{y}, \]

with \( \mathbf{k}_T = \hat{\mathbf{e}}_x k_x + \hat{\mathbf{e}}_y k_y \), are the spatiotemporal frequency spectra of the initial field vectors. If the initial time dependence of the field vectors on the plane \( z = z_0 \) is such that both field vectors \( \mathbf{E}_0(\mathbf{r}_T, t) \) and \( \mathbf{B}_0(\mathbf{r}_T, t) \) vanish for all \( t < t_0 \), then the temporal frequency transforms appearing in Eqs. (1.3) are both Laplace transformations and the contour \( C_1 \), appearing in Eqs. (1.2) is then the straight-line path \( \omega = \omega^0 + i\alpha \), with \( \alpha \) being greater than the abscissa of absolute convergence\(13\) for the initial time evolution of...
the field; if not, then they are both Fourier transformations. In addition,

$$\mathbf{k}^* = \hat{x}k_x + \hat{y}k_y + \hat{z}y$$  \hspace{1cm} (1.4)

is the complex wave vector for propagation into the positive half-space $z \geq z_0$, with the associated complex wave number

$$\tilde{k}(\omega) = (\mathbf{k}^* \cdot \mathbf{k}^*)^{1/2} = \frac{\omega}{c} n(\omega),$$ \hspace{1cm} (1.5)

where $c$ is the speed of light in vacuum and

$$n(\omega) = \left[\frac{1}{c} \right] \left[ \mu_0 \varepsilon(\omega) \right]^{1/2}$$ \hspace{1cm} (1.6)

is the complex index of refraction of the dispersive medium. Both Gaussian (cgs) and MKS units are employed here through the use of an appropriate conversion factor that appears in double brackets [[ ]] in any affected equation; if this factor is included in the equation it is then in cgs units, provided that one also sets both $\varepsilon_0$ and $\mu_0$ to unity, while if this factor is omitted the equation is in MKS units. The quantity $\varepsilon(\omega)$ appearing in Eq. (1.6) is the complex permittivity of the dispersive medium, given by

$$\varepsilon(\omega) = \varepsilon(\omega) + i\left[\frac{4\pi}{\omega} \right] \frac{\sigma(\omega)}{\omega}.$$ \hspace{1cm} (1.7)

The $z$ component of the complex wave vector appearing in Eq. (1.4) is defined by the principal branch of the expression

$$\gamma(\omega) = [\tilde{k}^2(\omega) - k_T^2]^{1/2}$$ \hspace{1cm} (1.8)

with $k_T^2 = k_T \cdot k_T = k_x^2 + k_y^2$.

Maxwell's equations require that the spatiotemporal spectra of the electric- and the magnetic-field vectors at the plane $z = z_0$ be related by

$$E_0(k_T, \omega) = -\frac{[c]}{\omega \mu_0 \varepsilon(\omega)} \mathbf{k}^* \times B_0(k_T, \omega),$$ \hspace{1cm} (1.9a)

$$B_0(k_T, \omega) = \frac{[c]}{\omega} \mathbf{k}^* \times E_0(k_T, \omega),$$ \hspace{1cm} (1.9b)

from which it is seen that the transversality condition

$$\mathbf{k}^* \cdot E_0(k_T, \omega) = \mathbf{k}^* \cdot B_0(k_T, \omega) = 0$$ \hspace{1cm} (1.10)

is satisfied. Equations (1.9) and (1.10) are precisely the relations between both field vectors and the wave vector for a time-harmonic electromagnetic plane-wave field in a homogeneous, isotropic, locally linear, temporally dispersive, and conducting medium. The pair of integrands appearing in the angular spectrum representation [Eqs. (1.2)], given by

$$E_0(k_T, \omega) \exp[i(\mathbf{k}^* \cdot \mathbf{r} - \omega t)] = E_0(k_T, \omega) \exp[i(k_Tx + k_yy + \gamma(z - z_0) - \omega t)],$$

$$B_0(k_T, \omega) \exp[i(\mathbf{k}^* \cdot \mathbf{r} - \omega t)] = B_0(k_T, \omega) \exp[i(k_Tx + k_yy + \gamma(z - z_0) - \omega t)],$$

is then seen to correspond to a harmonic electromagnetic plane-wave field that is propagating away from the plane $z = z_0$ at each angular frequency $\gamma$ and wave vector $\mathbf{k}^* = k_T + \hat{z}y$ that is present in the initial spectra of the field vectors at that plane.

As a final point here, note that either of the boundary values given in Eqs. (1.9) for the electric- and the magnetic-field vectors may be expressed in terms of the spectrum of the current source $j_0(\mathbf{r}, t)$ that resides in the region $z < z_0$ as

$$E_0(k_T, \omega) = \frac{[4\pi]}{2 \omega \mu_0 \varepsilon(\omega) \gamma(\omega)} \mathbf{k}^* \times [\mathbf{j}_0(\mathbf{k}^*, \omega)] \times \exp[i\gamma(\omega)z_0],$$ \hspace{1cm} (1.11a)

$$B_0(k_T, \omega) = -\frac{[4\pi]}{2 \gamma(\omega)} \mathbf{k}^* \times \mathbf{j}_0(\mathbf{k}^*, \omega) \times \exp[i\gamma(\omega)z_0].$$ \hspace{1cm} (1.11b)

Here

$$\mathbf{j}_0(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} j_0(\mathbf{r}, t) \exp[-i(\mathbf{k} \cdot \mathbf{r} - \omega t)] d^3r$$ \hspace{1cm} (1.12)

is the spatiotemporal Fourier-Laplace transform of the prescribed current source $j_0(\mathbf{r}, t)$. With these relations the analysis of the polarization properties of the radiated field in the half-space $z > z_0$ may be expressed in terms of the spectrum of the current source.

2. POLARIZATION ELLIPSE FOR THE COMPLEX FIELD VECTORS

From Eqs. (1.2) and (1.3) the angular spectrum of plane waves representation of the freely propagating electromagnetic field throughout the half-space $z \geq z_0$ may be
then Eq. (2.7) becomes
\[ \tan(2\varphi) = \tan(2\beta) \cos \gamma, \] (2.9)
and the principal semiaxes of the polarization ellipse at the observation point \( \mathbf{r}_0 \) are given by
\[ a^2 = \frac{1}{2} (p^2 + q^2) - \left( (p^2 - q^2)^2 + 4(p \cdot q)^2 \right)^{1/2}, \] (2.10a)
\[ b^2 = \frac{1}{2} (p^2 + q^2) - \left( (p^2 - q^2)^2 + 4(p \cdot q)^2 \right)^{1/2}, \] (2.10b)

Finally, the angle \( \Psi \), which the major axis makes with the vector \( \mathbf{p} \), is given by
\[ \tan \Psi = \frac{b}{a} \tan \varphi, \] (2.11)
as illustrated in Fig. 2. If the sign of the scalar triple product \( [\mathbf{a}, \mathbf{b}, \mathbf{V}_c] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{V}_c \) is positive, the polarization is left handed, while if the sign is negative the polarization is right handed.

Consider now the relation between the initial and the propagated polarization properties of the electromagnetic field in the half-space \( z \geq z_0 \). From Eqs. (2.2) and (2.6) the complex representation of the polarization ellipse for the propagated field vectors is seen to be given by
\[ \mathbf{E}(\mathbf{r}', \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathbf{E}_0(\mathbf{k}_T, \omega) \exp(i\mathbf{k}_T \cdot \mathbf{r}') d\mathbf{k}_T \cdot d\omega, \] (2.12a)
\[ \mathbf{B}(\mathbf{r}', \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathbf{B}_0(\mathbf{k}_T, \omega) \exp(i\mathbf{k}_T \cdot \mathbf{r}') d\mathbf{k}_T \cdot d\omega, \] (2.12b)
where \( \mathbf{r}' = \mathbf{r} + \mathbf{k}_T \cdot (z - z_0) \), \( z \geq z_0 \), is the position vector of the field observation point. Note that, with the relations given in Eqs. (1.9), the above pair of expressions may be expressed solely in terms of either \( \mathbf{E}_0(\mathbf{k}_T, \omega) \) or \( \mathbf{B}_0(\mathbf{k}_T, \omega) \), if desired. Here
\[ \mathbf{E}_0(\mathbf{k}_T, \omega) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) \exp(-i\mathbf{k}_T \cdot \mathbf{r}) d\mathbf{k}_T \cdot d\omega, \] (2.13a)
\[ \mathbf{B}_0(\mathbf{k}_T, \omega) = \int_{-\infty}^{\infty} \mathbf{B}(\mathbf{r}, \omega) \exp(-i\mathbf{k}_T \cdot \mathbf{r}) d\mathbf{k}_T \cdot d\omega, \] (2.13b)

Fig. 2. Polarization ellipse for the monochromatic complex field vector \( \mathbf{V}(\mathbf{r}, t) \) at a fixed point in space.
where \( \mathbf{k}_T = \hat{\mathbf{i}}_x k_x + \hat{\mathbf{i}}_y k_y \) and \( \mathbf{r}_T = \hat{\mathbf{i}}_x x' + \hat{\mathbf{i}}_y y' \), with

\[
\tilde{E}_0(\mathbf{r}_T, \omega) = \int_{-\infty}^{\infty} \mathcal{E}_0(\mathbf{r}_T, t) \exp(i\omega t) dt, \quad (2.14a)
\]

\[
\tilde{B}_0(\mathbf{r}_T, \omega) = \int_{-\infty}^{\infty} \mathcal{B}_0(\mathbf{r}_T, t) \exp(i\omega t) dt. \quad (2.14b)
\]

As in Eqs. (2.3), the initial field vectors on the plane \( z = z_0 \) may be written as

\[
\mathcal{E}_0(\mathbf{r}_T, t) = \frac{1}{\pi} \Re \left\{ \int_{C_\Gamma} V_e^{(0)}(\mathbf{r}_T, t; \omega) d\omega \right\}, \quad (2.15a)
\]

\[
\mathcal{B}_0(\mathbf{r}_T, t) = \frac{1}{\pi} \Re \left\{ \int_{C_\Gamma} V_m^{(0)}(\mathbf{r}_T, t; \omega) d\omega \right\}, \quad (2.15b)
\]

where

\[
V_e^{(0)}(\mathbf{r}_T, t; \omega) = \tilde{E}_e(\mathbf{r}_T, \omega) \exp(-i\omega t) = [p_e^{(0)}(\mathbf{r}_T, \omega) + i q_e^{(0)}(\mathbf{r}_T, \omega)] \exp(-i\omega t), \quad (2.16a)
\]

\[
V_m^{(0)}(\mathbf{r}_T, t; \omega) = \tilde{B}_m(\mathbf{r}_T, \omega) \exp(-i\omega t) = [p_m^{(0)}(\mathbf{r}_T, \omega) + i q_m^{(0)}(\mathbf{r}_T, \omega)] \exp(-i\omega t), \quad (2.16b)
\]

are complex vectors that describe the spatial properties of each monochromatic field component in the initial field. With the complex representation given in Eq. (2.6) for the polarization ellipse, one then has that

\[
\tilde{E}_0(\mathbf{r}_T, \omega) = p_e^{(0)}(\mathbf{r}_T, \omega) + i q_e^{(0)}(\mathbf{r}_T, \omega) = [a_e^{(0)}(\mathbf{r}_T, \omega) + ib_e^{(0)}(\mathbf{r}_T, \omega)] \exp[i\varphi_e^{(0)}(\mathbf{r}_T, \omega)], \quad (2.17a)
\]

\[
\tilde{B}_0(\mathbf{r}_T, \omega) = p_m^{(0)}(\mathbf{r}_T, \omega) + i q_m^{(0)}(\mathbf{r}_T, \omega) = [a_m^{(0)}(\mathbf{r}_T, \omega) + ib_m^{(0)}(\mathbf{r}_T, \omega)] \exp[i\varphi_m^{(0)}(\mathbf{r}_T, \omega)]. \quad (2.17b)
\]

Substitution of Eqs. (2.13) with Eqs. (2.17) into Eqs. (2.12) then yields the following general expression that is applicable to either the electric or the magnetic polarization ellipse:

\[
[a(r^+, \omega) + ib(r^+, \omega)] \exp[i\varphi(r^+, \omega)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ a_0(\mathbf{r}_T, \omega) + ib_0(\mathbf{r}_T, \omega) \right] \exp[i\varphi_0(\mathbf{r}_T, \omega)]
\]

\[
\times h(x - x', y - y' + \Delta z, \omega) dx' dy', \quad (2.18)
\]

where \( a_0 = a_e^{(0)} \) and \( b_0 = b_m^{(0)} \) for the electric polarization ellipse while \( a_0 = a_m^{(0)} \) and \( b_0 = b_e^{(0)} \) for the magnetic polarization ellipse. Here

\[
h(x - x', y - y' + \Delta z, \omega)\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(k_e \cdot \mathbf{r} + k_m \cdot \mathbf{r}_T)] dk_e dk_m \]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(k_e x - x') + k_m (y - y') + \gamma \Delta z)] dk_e dk_m, \quad (2.19)
\]

is the spatial impulse response function for the normal propagation distance \( \Delta z = z - z_0 \) in the dispersive medium at the angular frequency \( \omega \). The polarization properties of each monochromatic component in the propagated field are then seen to be given by the convolution of the initial polarization behavior on the plane \( z = z_0 \) with the spatial impulse response function for the normal propagation distance \( \Delta z \) in the dispersive medium at the angular frequency \( \omega \).

As a useful illustration, consider a uniformly polarized field vector over the plane \( z = z_0 \), in which case it is required that

\[
[a_0(\mathbf{r}_T, \omega) + ib_0(\mathbf{r}_T, \omega)] \exp[i\varphi_0(\mathbf{r}_T, \omega)]
\]

\[
= W_0(\mathbf{r}_T, \omega)(a_0 + ib_0) \exp(i\varphi_0), \quad (2.20)
\]

where \( a_0 \) and \( b_0 \) are fixed vectors and \( \varphi_0 \) is a constant. The only spatial variation in the particular field vector at the plane \( z = z_0 \) appears in the field amplitude through the scalar function \( W_0(\mathbf{r}_T, \omega) \). With this substitution, Eq. (2.18) yields

\[
[a(r^+, \omega) + ib(r^+, \omega)] \exp[i\varphi(r^+, \omega)]
\]

\[
= W_0(r^+, \omega)(a_0 + ib_0) \exp(i\varphi_0), \quad (2.21)
\]

where

\[
W(r^+, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_0(\mathbf{r}_T, \omega)
\]

\[
\times h(x - x', y - y' + \Delta z, \omega) dx' dy', \quad (2.22)
\]

and the polarization state for this field vector remains unchanged throughout the half-space \( z \geq z_0 \). Note that the condition given in Eq. (2.20) requires that the vectors \( a_0 \) and \( b_0 \) and the phase constant \( \varphi_0 \) all be independent of the angular frequency \( \omega \). If this is not the case the polarization state of the field vector will evolve with time. In general, the condition given in Eq. (2.20) is not satisfied, so the state of polarization must refer to the electromagnetic-field behavior at a particular point in space and, in general, varies from point to point in the field.\(^{1}\) In addition, at each point in space the state of polarization will, in general, vary with time.

### 3. RELATION BETWEEN THE ELECTRIC AND THE MAGNETIC POLARIZATION ELLIPSES

From Eqs. (2.2) and (2.6) the complex representation of the polarization ellipses for the temporal frequency spectra of the electric- and the magnetic-field vectors are given, respectively, by

\[
\tilde{E}(\mathbf{r}, \omega) = p_e(\mathbf{r}, \omega) + i q_e(\mathbf{r}, \omega) = e(\mathbf{r}, \omega) \exp[i\varphi_e(\mathbf{r}, \omega)], \quad (3.1a)
\]

\[
\tilde{B}(\mathbf{r}, \omega) = p_m(\mathbf{r}, \omega) + i q_m(\mathbf{r}, \omega) = b(\mathbf{r}, \omega) \exp[i\varphi_m(\mathbf{r}, \omega)], \quad (3.1b)
\]

where the complex vectors defined by

\[
e(\mathbf{r}, \omega) = a_e(\mathbf{r}, \omega) + ib_e(\mathbf{r}, \omega), \quad (3.2a)
\]

\[
b(\mathbf{r}, \omega) = a_m(\mathbf{r}, \omega) + ib_m(\mathbf{r}, \omega) \quad (3.2b)
\]
have been introduced here for notational convenience. Throughout the half-space \( z \geq z_0 \) the temporal frequency spectra of the field vectors satisfy the field equations

\[
\nabla \cdot \mathbf{E}(r, \omega) = 0, \\
\n\nabla \times \mathbf{E}(r, \omega) = \left[ \frac{1}{c} \right] i\omega \mathbf{B}(r, \omega), \\
\n\nabla \cdot \mathbf{B}(r, \omega) = 0, \\
\n\n\nabla \times \mathbf{B}(r, \omega) = -\left[ \frac{1}{c} \right] i\omega \mu_0 \epsilon_0 \mathbf{E}(r, \omega).
\]

(3.3a) (3.3b) (3.3c) (3.3d)

Substitution of the complex representation for the electric- and the magnetic-field vectors given in Eqs. (3.1) into the above divergence relations immediately yields the pair of relations

\[
\begin{align*}
\text{i}e(r, \omega) \cdot \nabla \epsilon(r, \omega) + \nabla \cdot \mathbf{E}(r, \omega) &= 0, \\
\text{i}b(r, \omega) \cdot \nabla \mu(r, \omega) + \nabla \cdot \mathbf{B}(r, \omega) &= 0,
\end{align*}
\]

(3.4a) (3.4b)

while substitution into the curl relations gives

\[
\begin{align*}
\mathbf{B}(r, \omega) &= \mathbf{b}(r, \omega)\exp(i\varphi_m(r, \omega)) \\
&= \frac{1}{\omega} \left[ \nabla \varphi_m(r, \omega) \times \mathbf{e}(r, \omega) - \text{i}\nabla \times \mathbf{e}(r, \omega) \right] \\
&\quad \times \exp[i\varphi_m(r, \omega)], \\
\mathbf{E}(r, \omega) &= \mathbf{e}(r, \omega)\exp(i\varphi_m(r, \omega)) \\
&= \frac{1}{\omega} \left[ -\nabla \varphi_m(r, \omega) \times \mathbf{b}(r, \omega) + \text{i}\nabla \times \mathbf{b}(r, \omega) \right] \\
&\quad \times \exp[i\varphi_m(r, \omega)].
\end{align*}
\]

(3.5a) (3.5b)

From this final pair of relations the polarization properties of one field vector can be directly determined from the polarization state of the other field vector at each point of space. For example, if the electric-field vector is linearly polarized with \( \mathbf{e}(r, \omega) = \mathbf{a}(r, \omega) \), where \( \mathbf{a} \) is a real-valued vector, then

\[
\mathbf{B}(r, \omega) = \mathbf{b}(r, \omega)\exp(i\varphi_m(r, \omega))
\]

\[
= \frac{1}{\omega} \left[ \nabla \varphi_m(r, \omega) \times \mathbf{a}(r, \omega) - \text{i}\nabla \times \mathbf{a}(r, \omega) \right] \exp[i\varphi_m(r, \omega)],
\]

and the magnetic-field vector will, in general, be elliptically polarized, provided that \( \mathbf{a} \) is spatially independent; if \( \mathbf{a} \) is independent of \( r \) so that the electric-field vector is uniformly linearly polarized throughout space, then \( \nabla \times \mathbf{a} = 0 \), and the magnetic-field vector is also linearly polarized.

As a consequence, the temporal frequency spectra of the electric- and the magnetic-field vectors are not, in general, instantaneously orthogonal to each other; indeed,

\[
\mathbf{E}(r, \omega) \cdot \mathbf{B}(r, \omega) = -\frac{1}{\omega} \text{i}\mathbf{e}(r, \omega) \\
\cdot \left[ \nabla \times \mathbf{e}(r, \omega) \right] \exp[2\varphi_m(r, \omega)],
\]

(3.6)

which does not, in general, vanish. As was stated in Section 2, the state of polarization refers strictly to the electromagnetic-field behavior at a particular point in space and, in general, varies from point to point in the field. Even if the electric-field vector is everywhere linearly polarized in the same direction, the magnetic-field vector will, in general, be elliptically polarized, as was first shown by Nisbet and Wolf. As a consequence, the temporal frequency spectra of the electric- and the magnetic-field vectors are not, in general, instantaneously orthogonal to each other. However, the long-time average of \( \mathbf{E}(r, \omega) \cdot \mathbf{B}(r, \omega) \), with \( \mathbf{E}(r, \omega) = \mathbf{E}(r, \omega)\exp(-\text{i}\omega t) \) and \( \mathbf{B}(r, \omega) = \mathbf{B}(r, \omega)\exp(-\text{i}\omega t) \), does vanish, since

\[
\langle \mathbf{E}(r, \omega) \cdot \mathbf{B}(r, \omega) \rangle = 0.
\]

(3.7)

The fact that the quantity \( \mathbf{e} \cdot (\nabla \times \mathbf{e}^*) \) used in the above proof is real valued directly follows from the vector differential identity \( \nabla \cdot (\nabla \times \mathbf{v}) = \mathbf{w} \cdot (\nabla \times \mathbf{v}) - \mathbf{v} \cdot (\nabla \times \mathbf{w}) \). With \( \mathbf{v} = \mathbf{e} \) and \( \mathbf{w} = \mathbf{e}^* \) and the fact that \( \mathbf{e} \times \mathbf{e}^* = 0 \), this identity yields the result \( \mathbf{e} \cdot (\nabla \times \mathbf{e}^*) = \mathbf{e}^* \cdot (\nabla \times \mathbf{e}) \), and consequently

\[
\begin{align*}
\mathbf{e} \cdot (\nabla \times \mathbf{e}^*) &= \frac{1}{2} [\mathbf{e} \cdot (\nabla \times \mathbf{e}^*) + \mathbf{e}^* \cdot (\nabla \times \mathbf{e})]
\end{align*}
\]

\[
= \frac{1}{2} [\mathbf{e} \cdot (\nabla \times \mathbf{e}^*) + [\mathbf{e} \cdot (\nabla \times \mathbf{e}^*)]^*]
\]

\[
= \Re[\mathbf{e} \cdot (\nabla \times \mathbf{e})^*].
\]

Thus the temporal spectra of the electric- and magnetic-field vectors are on the average mutually orthogonal.

As a summary of these results, illustrated in Fig. 3, it is first seen that the spatiotemporal frequency spectra \( \mathbf{E}(k, \omega) \) and \( \mathbf{B}(k, \omega) \) of the electric- and the magnetic-field vectors are mutually orthogonal. The temporal frequency spectra \( \mathbf{E}(r, t; \omega) \) and \( \mathbf{B}(r, t; \omega) \) of these field vectors are not, in general, orthogonal but are on the average mutually orthogonal. Finally, from Eqs. (3.1) and the above re-
sults it is seen that the electric-field \([\mathcal{E}(\mathbf{r}, t)]\) and the magnetic-field \([\mathcal{B}(\mathbf{r}, t)]\) vectors are not, in general, orthogonal and neither is their time average.

4. **UNIFORMLY POLARIZED FIELD**

For a uniformly polarized field in the electric-field vector, Eqs. (2.20) and (3.1) require that the initial field vectors on the plane \(z = z_0\) be given by

\[
\mathcal{E}_0(\mathbf{r}_T, t_0) = E_0(\mathbf{r}_T, t_0) (\mathbf{a}_e + i \mathbf{b}_m) \exp(\mathbf{i} \varphi_e),
\]

\[
\mathcal{B}_0(\mathbf{r}_T, t_0) = B_0(\mathbf{r}_T, t_0) [\mathbf{a}_m(\mathbf{r}_T, t_0) + i \mathbf{b}_n(\mathbf{r}_T, t_0)]
\times \exp(\mathbf{i} \varphi_m(\mathbf{r}_T, t_0)),
\]

where \(\mathbf{a}_e\) and \(\mathbf{b}_m\) are fixed vectors and \(\varphi_e\) is a scalar constant. These vectors are not independent and must be oriented such that Eqs. (3.5) are satisfied. In particular, Eq. (3.5a) requires that

\[
B_0(\mathbf{r}_T, t_0) [\mathbf{a}_m(\mathbf{r}_T, t_0) + i \mathbf{b}_n(\mathbf{r}_T, t_0)] \exp(\mathbf{i} \varphi_m(\mathbf{r}_T, t_0)) = -i \frac{[\mathcal{C}]}{\omega} [\mathcal{V} \mathcal{E}_0(\mathbf{r}_T, t_0)] \times (\mathbf{a}_e + i \mathbf{b}_m) \exp(\mathbf{i} \varphi_e),
\]

from which it is seen that the temporal frequency spectra of the electric- and the magnetic-field vectors of a completely uniformly polarized field are orthogonal. Although this is not necessary, the complex vectors \((\mathbf{a}_e + i \mathbf{b}_m)\) and \((\mathbf{a}_m + i \mathbf{b}_n)\) may be chosen to be normalized so that

\[|\mathbf{a} + \mathbf{b}|^2 = a^2 + b^2 = 1.\]

The spatiotemporal frequency spectra of the initial field vectors on the plane \(z = z_0\) are given, with Eqs. (4.1a) and (4.1b), respectively, by

\[
\mathcal{E}_0(\mathbf{k}_T, \omega) = (\mathbf{a}_e + i \mathbf{b}_m) \exp(\mathbf{i} \varphi_e) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}_0(\mathbf{r}_T, t_0) \exp(-i \mathbf{k}_T \cdot \mathbf{r}_T) d\mathbf{x} d\mathbf{y}
\times \exp(\mathbf{i} \varphi_m(\mathbf{r}_T, t_0)) \mathcal{B}_0(\mathbf{r}_T, t_0) \exp(-i \mathbf{k}_T \cdot \mathbf{r}_T) d\mathbf{x} d\mathbf{y},
\]

\[\mathcal{B}_0(\mathbf{k}_T, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{B}_0(\mathbf{r}_T, t_0) \exp(-i \mathbf{k}_T \cdot \mathbf{r}_T) d\mathbf{x} d\mathbf{y}.
\]

In addition, the space–time forms of the initial field vectors are given by

\[
\mathcal{E}(\mathbf{r}_T, t) = \frac{1}{\pi} \Re\{\mathcal{E}(\mathbf{r}_T, t)\}
\times \exp(-\mathbf{i} \omega t) d\omega,
\]

\[
\mathcal{B}(\mathbf{r}_T, t) = \frac{1}{\pi} \Re\{\mathcal{B}(\mathbf{r}_T, t)\}
\times \exp(\mathbf{i} \varphi_m(\mathbf{r}_T, t_0)) \mathcal{B}_0(\mathbf{r}_T, t_0) \exp(-\mathbf{i} \omega t) d\omega
\times \frac{[\mathcal{C}]}{\pi} \Re\{\mathcal{E}(\mathbf{r}_T, t)\} \times \int_{-\infty}^{\infty} \frac{1}{\omega} \mathcal{E}_0(\mathbf{r}_T, t_0) \exp(-\mathbf{i} \omega t) d\omega.
\]

From Eqs. (1.2) with Eqs. (4.3) the propagated field vectors are found as

\[
\mathcal{E}(\mathbf{r}, z, t) = \frac{1}{4\pi^2} \Re\{\mathcal{E}(\mathbf{r}, z, t)\}
\times \exp(\mathbf{i} \varphi_m) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{B}_0(\mathbf{k}_T, \omega) \exp(-\mathbf{i} \mathbf{k}_T \cdot \mathbf{r}_T) d\mathbf{k}_T d\omega,
\]

\[
\mathcal{B}(\mathbf{r}, z, t) = \frac{-[\mathcal{C}]}{4\pi^2} \Re\{\mathcal{B}(\mathbf{r}, z, t)\}
\times \exp(\mathbf{i} \varphi_m) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}_0(\mathbf{k}_T, \omega) \exp(-\mathbf{i} \mathbf{k}_T \cdot \mathbf{r}_T) d\mathbf{k}_T d\omega.
\]
and hence
\[
(a_n + ib_m)\exp(i\varphi_n)B_0(k_T, \omega) = -\left[\frac{1}{\omega}\right]k^1(\omega)E_0(k_T, \omega)(a_n + ib_n) \times \hat{s} \exp(i\varphi_n),
\]
(4.9)
where \(\hat{s} = \hat{i}_x p + \hat{i}_y q + \hat{i}_z m\), with \(m = (1 - p^2 - q^2)^{1/2}\).

Note that this result also follows from Eq. (1.9b), viz.,

\[
B_0(k_T, \omega) = \left[\frac{1}{\omega}\right]k^* \times E_0(k_T, \omega),
\]

with \(E_0(k_T, \omega)\) given by Eq. (4.3a) and \(B_0(k_T, \omega)\) given by the spatial transform of Eq. (4.7b). The orientation of the magnetic polarization ellipse is then seen to depend on the orientation of the electric polarization ellipse through the spatial transform of Eq. (4.7b). The orientation of the electric- and the magnetic-field vectors (\(r, t\)) are independent of the \(y\) coordinate. As another example, let the uniformly polarized electric field be linearly polarized along the \(x\) axis so that \(a_n = \hat{i}_x, b_n = 0\), and \(\varphi_n = 0\), then

\[
(a_n + ib_m)\exp(i\varphi_n) = \left(\hat{i}_x p + \hat{i}_y q + \hat{i}_z (1 - p^2 - q^2)^{1/2}\right) \times \hat{i}_x,
\]

(4.9)
from which it is seen that there are two possibilities to maintain the requirement that the magnetic field be uniformly polarized. Either \(q = q_0\) is fixed, in which case \(p\) may be permitted to vary such that either \(p^2 \leq 1 - q_0^2\) or \(p^2 \geq 1 - q_0^2\), or else \(q\) is permitted to vary, in which case \(p\) must vary in such a fashion that \(m = 0\), i.e., that \(p^2 = 1 - q^2\). The situation that results when \(m = 0\) precludes propagation into the half-space \(\Delta z > 0\) and so is not of interest here. Hence \(q = q_0\), and the field itself must then be independent of the \(y\) coordinate. As another example, let the uniformly polarized electric field be circularly polarized in the \(xy\) plane so that \(a_n = \hat{i}_x, b_n = \hat{i}_y,\) and \(\varphi_n = 0\), in which case

\[
(a_n + ib_m)\exp(i\varphi_n) = \left(\hat{i}_x p + \hat{i}_y q + \hat{i}_z (1 - p^2 - q^2)^{1/2}\right) \times \hat{i}_x + \hat{i}_y,
\]

\[
\hat{i}_z (1 - p^2 - q^2)^{1/2} - \hat{i}_y q,
\]
and \(p\) and \(q\) must both be fixed (i.e., \(p = p_0, q = q_0\)) for the magnetic-field-polarization ellipse to be uniform. Since \(a_n^2 = 1 - p_0^2\) and \(b_m^2 = 1 - q_0^2\), the magnetic field is elliptically polarized in general, and it is circularly polarized if \(p_0 = q_0\). In either case, the field itself is independent of both the \(x\) and the \(y\) coordinates. In general, the requirement that both field vectors be uniformly polarized demands that the field be independent of some coordinate direction.

5. SUMMARY

The angular spectrum of plane waves representation of the freely propagating electromagnetic field in the half-space \(z \geq z_0\) of a homogeneous, isotropic, locally linear, temporally dispersive medium has been used to describe the polarization properties of the field throughout the half-space \(z > z_0\) in terms of the polarization behavior of either field vector on the plane at \(z = z_0\). Although the spatiotemporal frequency spectra of the electric- and the magnetic-field vectors are mutually orthogonal, the temporal frequency spectra of these field vectors are not, in general, orthogonal at any fixed point in the half-space but are on the average mutually orthogonal. Furthermore, the electric- and the magnetic-field vectors \(E(r, t)\) and \(B(r, t)\) are not, in general, orthogonal, and neither is their time average. Finally, the requirement that both field vectors be uniformly polarized demands that the field be independent of some coordinate direction. It is clear that these results provide a useful framework for the description of the evolutionary properties of pulsed electromagnetic beam fields in a temporally dispersive medium.

ACKNOWLEDGMENT

The research presented in this paper was supported by the U.S. Air Force Office of Scientific Research under contract FA9550-89-C-0057.

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