Asymptotic description of pulsed ultrawideband electromagnetic beam field propagation in dispersive, attenuative media

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The asymptotic description of the coupled spatial and temporal evolution of a pulsed ultrawideband electromagnetic beam field as it propagates through a dispersive, attenuative material that occupies the half-space \( z \gg z_0 \) is obtained from the angular spectrum of plane waves representation. This angular-spectrum representation expresses the wave field as a superposition of both homogeneous and inhomogeneous plane waves. The paraxial approximation of the spatial part of this representation for nontruncated beam fields results in a description that explicitly displays the temporal evolution of the pulsed-beam field through a single-contour integral that is of the same form as that obtained for a pulsed plane-wave field propagating in the positive \( z \) direction in a lossy, dispersive medium. The accuracy of this paraxial approximation is shown to improve as the material’s attenuation increases.

1. INTRODUCTION

In any homogeneous, isotropic, locally linear medium the angular spectrum of plane-waves representation expresses any freely propagating electromagnetic field that propagates into the half-space \( z \gg z_0 \) as a superposition of both homogeneous and inhomogeneous plane waves. The inhomogeneous plane-wave components are unimportant in many instances when the material is lossless because they become evanescent in the propagation direction. Because of this evanescence, an electromagnetic beam field in a lossless medium is typically defined, in part, by the requirement that it can be represented by an angular spectrum that contains only homogeneous waves. As the homogeneous waves do not attenuate in such a lossless medium, this condition ensures that all the angular-spectrum components of the beam field remain in their original proportions throughout the half-space \( z \gg z_0 \).

Wave fields that do not contain any evanescent components in their angular-spectrum representation are known as source-free wave fields. The general properties of such source-free wave fields in lossless, nondispersive media were described by Sherman in an analysis that is now classical. This analysis was extended to the important case of source-free wave fields in temporally dispersive, absorptive media. In this case no convenient separation exists between lossless homogeneous and lossy evanescent fields because, when the material is attenuative, all the plane-wave components become inhomogeneous with the exception of a single \( z \)-directed homogeneous plane-wave component. The analysis presented in this paper shows that the accuracy of the paraxial approximation, which neglects those inhomogeneous plane-wave components whose direction cosines lie outside a subunity radius circle, improves as the material absorption increases.

2. ANGULAR-SPECTRUM REPRESENTATION OF THE FREELY PROPAGATING WAVE FIELD

Consider the evolution of a freely propagating electromagnetic field in the half-space \( z \gg z_0 > Z \) that is occupied by a homogeneous, isotropic, locally linear, temporally dispersive medium. The term freely propagating is used to indicate that the field source resides entirely in the region \( z \ll Z \). It is unnecessary to know what this source is, provided that the tangential components of either of the field vectors \( \mathbf{E}(r_T, z_0, t) = E_0(r_T, t) \) or \( \mathbf{B}(r_T, z_0, t) = B_0(r_T, t) \) are known functions of time and the transverse position vector \( r_T = \hat{l}_x x + \hat{l}_y y \) in the plane \( z = z_0 \). It is assumed that the two-dimensional spatial Fourier transform in the transverse coordinates \((x, y)\) as well as the temporal Fourier–Laplace transform of each field vector at the plane \( z = z_0 \) exist, as given by the pair of relations

\[
\tilde{U}_0(k_T, \omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ U_0(r_T, t) \times \exp[-i(k_T \cdot r_T - \omega t)],
\]

\[
U_0(r_T, t) = \left\{ \frac{1}{4\pi^2} \right\}^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x dk_y \tilde{U}_0(k_T, \omega) \times \exp[i(k_T \cdot r_T - \omega t)] \right\},
\]

where \( k_T = \hat{l}_x k_x + \hat{l}_y k_y \) is the transverse wave vector. Here \( U_0(r_T, t) \) represents either the electric \( \mathbf{E}_0(r_T, t) \) or the magnetic \( \mathbf{B}_0(r_T, t) \) field; \( \tilde{U}_0(k_T, \omega) \) then represents the corresponding Fourier–Laplace transform \( \mathbf{E}_0(k_T, \omega) \) or \( \mathbf{B}_0(k_T, \omega) \) of the electric or the magnetic field vector.
respectively. If the initial time dependence of either field vector $\mathbf{E}_0(r_T, t)$ or $\mathbf{B}_0(r_T, t)$ at the plane $z = z_0$ is such that it vanishes for all $t < t_0$ for some finite value of $t_0$, then the transforms that appear in Eqs. (1) are Laplace transformations, and the contour of integration $C_z$ is the straight-line path $\omega = \omega' + ia$, with $a$ greater than the abscissa of absolute convergence for the initial time behavior of the field and with $\omega' = \Im(\omega)$ varying from 0 to $\pm \infty$. If these conditions are not met they are Fourier transformations.

**A. Angular Spectrum of Plane Waves Representation**

The freely propagating electromagnetic field throughout the half-space $z \geq z_0$ is given by the angular spectrum of plane waves representation:

$$U(r_T, z, t) = \left(1/4\pi^3\right) \Re \left( \int_{C_z} d\omega \int_0^\infty dk_x dk_y \tilde{U}_0(k_T, \omega) \right.$$

$$\left. \times \exp\left(i[k_T \cdot r_T + \gamma(\omega)z - \omega t]\right) \right)$$  \hspace{1cm} (2)

for both the electric and the magnetic field vectors. The spatiotemporal spectra of the electric and the magnetic field vectors at the plane $z = z_0$ satisfy the transversality conditions:

$$\mathbf{E}_0(k_T, \omega) = -\left(\|c\|/\omega \mu \varepsilon_0(\omega)\right) \tilde{\mathbf{E}}^+(\omega) \times \mathbf{B}_0(k_T, \omega)$$ \hspace{1cm} (3a)

$$\mathbf{B}_0(k_T, \omega) = \left(\|c\|/\omega \varepsilon_0(\omega)\right) \mathbf{E}^+(\omega) \times \mathbf{E}_0(k_T, \omega)$$ \hspace{1cm} (3b)

so $\tilde{\mathbf{E}}^+(\omega) \cdot \mathbf{E}_0(k_T, \omega) = \tilde{\mathbf{B}}^+(\omega) \cdot \mathbf{B}_0(k_T, \omega) = 0$. Here

$$\tilde{k}^+(\omega) = \mathbf{i}_x k_x + \mathbf{i}_y k_y + \mathbf{i}_z \gamma(\omega)$$ \hspace{1cm} (4)

is the complex wave vector for propagation into the positive half-space $z \geq z_0$, with the associated complex wave number

$$k(\omega) = (\tilde{k}^+ \cdot \tilde{k}^+)^{1/2} = \|1/c\| \omega (\mu \varepsilon_0(\omega))^{1/2}$$ \hspace{1cm} (5)

and $\gamma(\omega)$ is defined as the principal branch of the expression

$$\gamma(\omega) = (k^2(\omega) - k_T^2)^{1/2}$$ \hspace{1cm} (6)

with $k_T^2 = k_x^2 + k_y^2$. This branch choice results in exponential attenuation with increasing $\Delta z \equiv 0$ for each plane-wave spectral component. Here

$$\varepsilon_0(\omega) = \varepsilon(\omega) + i\pi \|\sigma(\omega)\|/\omega$$ \hspace{1cm} (7)

is the complex permittivity of the dispersive medium with a frequency-dependent dielectric permittivity, $\varepsilon(\omega) = \varepsilon_r(\omega) + i\varepsilon_i(\omega)$, and an electric conductivity, $\sigma(\omega) = \sigma_r(\omega) + i\sigma_i(\omega)$, whose real and imaginary parts are each connected through the appropriate dispersion relation. The integrand

$$\tilde{U}_0(k_T, \omega) \exp\left[i(k_T \cdot r_T + \gamma(\omega)z - \omega t)\right]$$

is $\tilde{U}_0(k_T, \omega)$ at each angular frequency $\omega$ and transverse wave vector $k_T = \mathbf{i}_x k_x + \mathbf{i}_y k_y$ that is present in the initial spectral field vectors $\{\mathbf{E}_0(k_T, \omega), \mathbf{B}_0(k_T, \omega)\}$ at that plane with one distinction: The wave-vector components $k_x$ and $k_y$ are real valued and independent of $\omega$, whereas $\gamma(\omega) = (k^2(\omega) - k_x^2 - k_y^2)^{1/2}$ is, in general, complex valued. Consequently, each plane-wave component that appears in the angular-spectrum representation [Eq. (2)] is attenuated in the $z$ direction alone.

**B. Real Direction-Cosine Form of the Angular-Spectrum Representation**

The plane-wave components that appear in Eq. (2) can be expressed in the geometric forms:

$k_x = k(\omega)p$, $k_y = k(\omega)q$, $\gamma(\omega) = k(\omega)m$, \hspace{1cm} (9)

with both $p$ and $q$ real valued. Here $k(\omega)$ is the wave number in vacuum, $n(\omega)$ is the wave number in vacuum, and $\gamma(\omega)$ is the complex index of refraction of the dispersive medium, and

$$k(\omega) = \beta(\omega) + i\alpha(\omega) = k(\omega)\exp[i\psi(\omega)]$$ \hspace{1cm} (10)

with the plane-wave propagation $\beta(\omega) + \alpha(\omega) = \Re[k(\omega)]$ and attenuation $\alpha(\omega) = \Im[k(\omega)]$ factors.

With these substitutions, Eq. (6) yields

$$m^r(\omega) = \Re[m(\omega)] = 0, \hspace{1cm} m^i(\omega) = \Re[m(\omega)] = 0$$ \hspace{1cm} (12)

for all $\omega \in C_z$. Explicit expressions for both $m^r(\omega)$ and $m^i(\omega)$ are obtained from the square of Eq. (11) as

$$m^r(\omega) = \left[1/2 \left\{\cos[2\psi(\omega)] - (p^2 + q^2)\right\}ight.$$

$$\times \left.\left\{\cos[2\psi(\omega)] - (p^2 + q^2)\right\}ight]^{1/2} \frac{1}{\sin^2[2\psi(\omega)]^{1/2}}$$ \hspace{1cm} (13a)

provided that $\sin[2\psi(\omega)] \neq 0$, and as

$$m^i(\omega) = \frac{\sin[2\psi(\omega)]}{2m^r(\omega)}$$ \hspace{1cm} (13b)

provided that $m^r(\omega) \neq 0$. If $m^r(\omega) = 0$, then $\sin[2\psi(\omega)] = 0$, and either $\psi(\omega) = 0$, in which case $(p^2 + q^2) > 1$ and

$$m^r(\omega) = [(p^2 + q^2) - 1]^{1/2}, \hspace{1cm} (14)$$

for $\psi(\omega) = \pi/2$, in which case

$$m^r(\omega) = [(p^2 + q^2) + 1]^{1/2}, \hspace{1cm} (15)$$

for all values of $p$ and $q$. For the special case of a lossless medium, $\psi(\omega) = 0$ and $m^r(\omega) = m^i(\omega) = [1 - (p^2 + q^2) - 1]^{1/2}$.\hspace{1cm}
with the inhomogeneous plane-wave spectral components in the exterior region, \( R_\omega = \{(p, q) | (p^2 + q^2) > \cos 2\psi(\omega)\} \), is typically larger than that associated with the inhomogeneous plane-wave spectral components in the interior region, \( R_\omega = \{(p, q) | (p^2 + q^2) < \cos 2\psi(\omega)\} \).

With these results the angular spectrum of plane wave representation [Eq. (2)] for both the electric and the magnetic field vectors becomes

\[
U(r, t) = (1/4\pi^2)\Re\left(\int_{C_{\infty}} d\omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k^2(\omega) \widehat{U}_0(p, q, \omega) \exp[i(k(\omega)[px + qy + m(\omega)\Delta z])] dp dq \right),
\]

where \( \widehat{U}_0(p, q, \omega) = \widehat{U}_0(k_x, k_y, k_z / k(\omega), \omega) \).

3. PARAXIAL APPROXIMATION OF THE ANGULAR-SPECTRUM REPRESENTATION

In the paraxial approximation the complex direction cosine \( m(\omega) \) is approximated as

\[
m(\omega) = m'(\omega) = i\beta(\omega) + a(\omega),
\]

so the propagation kernel \( G(p, q, \omega) \) becomes

\[
G(p, q, \omega) = \exp[i(k(\omega)m(\omega)\Delta z)]
\]

\[
\times [\exp[-\bar{k}(\omega)\Delta z] \exp[-\bar{k}^*(\omega)\Delta z(p^2 + q^2)/2],
\]

where the superscript asterisk denotes complex conjugation. Limits on the accuracy of this approximation were recently published by Melamed and Felsen\(^{10,11}\) for pulsed-beam propagation in lossless dispersive media. Fortunately, the situation is simplified when material loss is present, as described in Subsections 3.A and 3.B.

A. Numerical Example

Consider the Rocard–Powles model\(^{12,13}\) of triply distilled water at 25 °C, where the causal function

\[
e(\omega) = \varepsilon_\infty + \sum_{j=1}^{2} \frac{a_j}{(1 - i\tau_j\omega)(1 - i\tau_j\omega)}
\]

accurately describes the complex-valued relative dielectric permittivity over the angular-frequency domain \( 0 < \omega < 1 \times 10^{13} \text{rad/s} \). Here \( \varepsilon_\infty = 2.1 \) denotes the large frequency limit of the relative dielectric permittivity, \( \tau_1 = 74.1 \) and \( \tau_2 = 2.90 \) are nondimensional scalar quantities describing the strength of each Rocard–Powles feature, \( \tau_1 = 8.44 \times 10^{-12} \text{s} \) and \( \tau_2 = 6.05 \times 10^{-14} \text{s} \) are the macroscopic Debye relaxation times for each feature, and \( \tau_1 = 4.93 \times 10^{-14} \text{s} \) and \( \tau_2 = 8.59 \times 10^{-15} \text{s} \) are the associated friction times introduced by the Rocard–Powles extension\(^{14}\) of the Debye model.\(^{15}\)

The resultant frequency dispersion of the real and the imaginary parts of the complex wave number \( \bar{k}(\omega) = \beta(\omega) + i\alpha(\omega) \) is depicted in Fig. 1. Note the peak in

![Fig. 1. Angular-frequency dependence of (a) the real and (b) the imaginary parts of the complex wave number \( \bar{k}(\omega) = \beta(\omega) + i\alpha(\omega) \) for the Rocard–Powles model of triply distilled water at 25 °C. The values marked with a cross (×) in each diagram indicate the real and the imaginary values at the angular frequencies of \( \omega_c = 2 \pi \times 10^7 \text{rad/s} \) in the HF, \( \omega_c = 2 \pi \times 10^9 \text{rad/s} \) in the EHF, and \( \omega_c = 2 \pi \times 10^{11} \text{rad/s} \) in the EHF regions of the electromagnetic spectrum that are used in several examples in this paper.](image-url)
the absorption at $\omega = 6 \times 10^{13} \text{rad/s}$, which is followed by a monotonic decrease in attenuation for larger frequencies. For angular frequencies greater than $\omega = 1 \times 10^{13} \text{rad/s}$, resonance-polarization effects begin to appear, and the rotational polarization model leading to Eq. (21) must be augmented by several Lorentz lines\(^{12}\) that add resonance features to the curves depicted in Fig. 1. This additional frequency-dependent structure is not necessary for the purpose of the present paper and so is neglected; however, its neglect does not limit the accuracy of the general results presented here.

At the lower frequency ($\omega_c = 2 \pi \times 10^7 \text{rad/s}$) indicated in Fig. 1 the absorption is small [$\alpha(\omega_c) = 4.66 \times 10^{-4} \text{m}^{-1}$] and $\cos(2\phi(\omega_c)) = 1.0000$ so that the complex direction cosine $m(\omega_c)$ is practically the same as that for an ideal lossless medium. This is evident from Fig. 2, which depicts the behavior of the real and the imaginary parts of $m(\omega_c)$ as functions of $p$ with $q = 0$. The dashed curves in Fig. 2 depict the behavior of the paraxial approximation given in expression (19). The real part of this paraxial approximation is seen to be accurate for $|p| \leq 0.6$, whereas the imaginary part, which is approximately zero for all real values of $p$, is accurate for all $|p| \leq 1$.

The above behavior is reflected in Fig. 3, which depicts the behavior of the magnitude and the phase of the propagation kernel $G(p, q, \omega_c)$, as well as in Fig. 4, which depicts the behavior of the real and the imaginary parts of $G(p, q, \omega_c)$ as functions of $p$ with $q = 0$ at the propagation distance $\Delta z = 0.1 \text{m}$. These results clearly demonstrate the well-known result that the paraxial approxima-

Fig. 2. (a) Real and (b) imaginary parts of the complex direction cosine $m(\omega_c) = \{\exp\{i2\phi(\omega_c)\} - (p^2 + q^2)^{1/2}\}$ at $\omega_c = 2 \pi \times 10^7 \text{rad/s}$ plotted as functions of $p$ with $q = 0$. The solid curves depict the exact behavior, whereas the dashed curve depicts the behavior obtained with the paraxial approximation.

Fig. 3. (a) Magnitude and (b) phase of the complex direction cosine $m(\omega_c) = \{\exp\{i2\phi(\omega_c)\} - (p^2 + q^2)^{1/2}\}$ at $\omega_c = 2 \pi \times 10^7 \text{rad/s}$ plotted as functions of $p$ with $q = 0$. The solid curves represent the exact behavior, whereas the dashed curve depicts the behavior obtained with the paraxial approximation.

Fig. 4. (a) Real and (b) imaginary parts of the propagation kernel $G(p, q, \omega_c) = \exp\{ik(\omega_c) m(\omega_c) \Delta z\}$ at $\omega_c = 2 \pi \times 10^7 \text{rad/s}$ with $\Delta z = 0.1 \text{m}$ plotted as functions of $p$ with $q = 0$. The solid curves depict the exact behavior, whereas the dashed curves depict the behavior obtained with the paraxial approximation.

Fig. 5. Same as is shown in Fig. 2 but with $\omega_c = 2 \pi \times 10^9 \text{rad/s}$. 
At the upper frequency ($\omega_c = 2\pi \times 10^{11}$ rad/s) indicated in Fig. 1 the absorption is high [$\alpha(\omega_c) = 4.27 \times 10^4$ m$^{-1}$] and $\cos[2\phi(\omega_c)] = 0.4615$. The complex direction cosine $m(\omega_c)$ is then a smoothed version of that for an ideal lossless medium, as can be seen from Fig. 8. The accuracy of the paraxial approximation has improved significantly over the previous two cases, particularly in the interior region defined by $(p^2 + q^2) \leq \cos[2\phi(\omega_c)]$, and remains reasonably accurate out to twice this value. This improved accuracy is reflected in the paraxial approximation of the propagation kernel $G(p, q, \omega_c)$, whose magnitude and phase are depicted in Fig. 9 and whose real and imaginary parts are depicted in Fig. 10 at $\Delta z = 0.1$ m. Because of the high value of the absorption at this frequency, the paraxial approximation of $G(p, q, \omega_c)$ is nearly identical to the exact behavior for all values of $p$ and $q$ for which $|G(p, q, \omega_c)|$ is essentially nonzero.

Finally, note that the accuracy of the paraxial approximation of the propagation kernel $G(p, q, \omega_c)$ increases as

At the intermediate frequency ($\omega_c = 2\pi \times 10^9$ rad/s) indicated in Fig. 1 the absorption is moderate [\(\alpha(\omega_c) = 4.65\) m$^{-1}$] and $\cos[2\phi(\omega_c)] = 0.9988$. The complex direction cosine $m(\omega_c)$ is then a slightly smoothed version of that for an ideal lossless medium, as can be seen from Fig. 5. The accuracy of the paraxial approximation is seen to be slightly improved over the low-loss case depicted in Fig. 2, and this slight improvement is reflected in the paraxial approximation of the propagation kernel $G(p, q, \omega_c)$, whose magnitude and phase are depicted in Fig. 6 and whose real and imaginary parts are depicted in Fig. 7 at $\Delta z = 0.1$ m. Note that both the amplitude and the phase of the paraxial approximation of $G(p, q, \omega_c)$ lose accuracy as $(p^2 + q^2)^{1/2}$ approaches unity from below in this moderate-absorption case, so the real and the imaginary parts of $G(p, q, \omega_c)$ exhibit nearly identical behavior.
the propagation distance increases. This property is illustrated in Figs. 7 (for $\Delta z = 0.1 \text{ m}$), 11 (for $\Delta z = 1.0 \text{ m}$) and 12 (for $\Delta z = 10.0 \text{ m}$) for the intermediate-frequency case ($\omega_c = 2\pi \times 10^9 \text{ rad/s}$) for which the absorption is moderate. As the propagation distance increases, both the real and the imaginary parts of the paraxial approximation of the propagation kernel improve in accuracy. At the largest propagation distance considered the error is almost exclusively due to a small $(p, q)$-dependent phase error, as can be seen from Fig. 12.

These results thus show that the paraxial approximation improves in accuracy as the material attenuation increases as well as when the propagation distance into the attenuative material increases. This trend indicates that the inhomogeneous plane-wave components in the exterior domain, $R_+ =\{(p, q) | p^2 + q^2 > \cos[2\psi(\omega)]\}$, become entirely negligible in comparison with the homogeneous and the inhomogeneous plane-wave components in the interior domain, $R_- =\{(p, q) | p^2 + q^2 < \cos[2\psi(\omega)]\}$, as the propagation distance $\Delta z$ typically exceeds a single absorption depth $z_d = \alpha^{-1}(\omega_c)$ at the input pulse-carrier frequency. This property is ideally suited to asymptotic methods of analysis wherein the asymptotic description increases in accuracy in the sense of Poincare$^{16}$ as the propagation distance $\Delta z$ increases above $z_d = \alpha^{-1}(\omega_c)$. Moreover, it is expected that, because of error cancellation on integration, the overall accuracy of the final result obtained from Eq. (18) could be even better. However, this improvement needs to be considered on a case-by-case basis for each type of wave field (e.g., Hermite–Gaussian or Laguerre–Gaussian beam fields).

B. Paraxial Approximation

Substitution of the paraxial approximation (20) of the propagation kernel into the angular-spectrum representation [Eq. (18)] yields

$$U(r, t) = \frac{1}{i\pi} \text{Re} \left( \int_{C_+} F(\mathbf{r}, \omega) \exp[i(\tilde{k}(\omega)\Delta z - \omega t)] d\omega \right),$$

(22)

where

$$\tilde{F}(\mathbf{r}, \omega) \equiv -i[\tilde{k}(\omega)/2\Delta z] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{U}_0(x', y', \omega) \times \exp[i(\tilde{k}(\omega)/2\Delta z)] \times [(x - x')^2 + (y - y')^2] dx' dy'$$

(23)

is one form of the well-known Fresnel–Kirchhoff diffraction integral. Here

$$\tilde{U}_0(x, y, \omega) = \int_{-\infty}^{\infty} U_0(x, y, t) \exp(i\omega t) dt$$

(24)

denotes the Fourier–Laplace transform of the initial field vector at the plane $z = z_0$. Because $\tilde{k}(\omega) = \beta(\omega) + i\alpha(\omega)$, the Fresnel–Kirchhoff diffraction integral (23) may be rewritten in the form

![Fig. 11. (a) Real and (b) imaginary parts of the propagation kernel $G(p, q, \omega_c)$ at $\omega_c = 2\pi \times 10^9 \text{ rad/s}$ with $\Delta z = 1.0 \text{ m}$ plotted as functions of $p$ with $q = 0$. The solid curves depict the exact behavior, whereas the dashed curves depict the behavior obtained with the paraxial approximation.](image1)

![Fig. 12. Same as is shown in Fig. 11 but with $\Delta z = 10.0 \text{ m}$.](image2)
\[ \tilde{F}(\mathbf{r}, \omega) \equiv \int [\hat{k}(\omega)/2\pi \Delta z] \int_{-\infty}^{\infty} \hat{U}_0(x', y', \omega) \times \exp\left[-\left(\alpha(\omega)/2\Delta z\right)(x-x')^2 + (y-y')^2\right] \exp\left[i\beta(\omega)/2\Delta z\right][x-x')^2 + (y-y')^2] \, dx' \, dy', \] 

which explicitly displays the manner in which the attenuation influences the beam field diffraction. Specifically, material attenuation decreases the effects of diffraction relative to the geometrical-optics contribution.

4. ASYMPTOTIC DESCRIPTION OF THE PARAXIAL APPROXIMATION OF THE ANGULAR-SPECTRUM REPRESENTATION

The angular-frequency contour integral appearing in Eq. (22), which can be rewritten in the form

\[ U(\mathbf{r}, t) = (1/\pi)R \int_{C} \tilde{F}(\mathbf{r}, \omega) \exp\left[(\Delta z/c)\phi(\omega, \theta)\right] d\omega, \] 

is ideally suited for asymptotic methods of analysis as \( \Delta z \to \infty \) when the spectral function \( \tilde{F}(\mathbf{r}, \omega) \) is represented by the paraxially approximate expression that is given in either expression (23) or (25). In this asymptotic limit the accuracy of the paraxial approximation in an attenuative medium increases, resulting in an approximation that fits naturally into the asymptotic description. Here

\[ \phi(\omega, \theta) = i(c/\Delta z)[\hat{k}(\omega)/\Delta z - i\omega t] = i\omega[n(\omega) - \theta] \] 

(27)
is a complex-phase function with \( \theta = ct/\Delta z \). This asymptotic description relies on the saddle points of \( \phi(\omega, \theta) \). Each saddle point is a solution of the saddle-point equation, \( \partial F/\partial \omega = 0 \), whose dynamical evolution in the complex \( \omega \) plane as the space–time parameter \( \theta \) varies describes the dynamical \( (z, t) \) evolution of the propagated pulse. For a given class of beam field (i.e., Hermite–Gaussian or Laguerre–Gaussian), the Fresnel–Kirchhoff integral appearing in expressions (23) and (25) may be evaluated directly, and this result may then be substituted into the contour integral appearing in Eq. (22), the resulting integration then being carried out by well-defined asymptotic methods. This detailed analysis is best accomplished in a case-by-case or a canonical approach and is the subject of considerable current and planned research. For purposes of brevity only a general description is presented here for the case of nontruncated beam fields.

Previously published results\(^{18}\) show that the complex-phase behavior \( \phi(\omega, \theta) \) for a general, causal dielectric in both the low-frequency region, \( |\omega| \ll \omega_0 \), below the lowest characteristic frequency of the dispersive medium and the high-frequency region, \( |\omega| \gg \omega_0 \), above the highest medium-resonance frequency results in a separation between two distinct classes of dielectrics: the Lorentz-type dielectric and the Debye-type dielectric. Here \( \theta = ct/\Delta z \) is a dimensionless space–time parameter that plays a central role throughout the asymptotic description.\(^{1,18–25}\) It provides a convenient nondimen-

\[ \text{sional time parameter at any fixed propagation distance } \Delta z \text{ into the dispersive medium such that the relative velocity } v/jc \text{ of a given feature of the pulse is given by } 1/\theta, \text{ where } \theta \text{ denotes the space–time point at which that feature occurs in the asymptotic description.} \]

For a Lorentz-type dielectric, both the distant and the near saddle points\(^{19,20}\) plus additional middle saddle points\(^{21}\) contribute to the asymptotic behavior of the propagated field, so the asymptotic description of the propagated field given in Eq. (26) that is produced by an input ultrawideband pulse can be expressed as\(^{1,18–21}\)

\[ U(r_T, z, t) = U_S(r_T, z, t) + U_B(r_T, z, t) \]

and

\[ U_i(r_T, z, t) \sim (c/2\pi \Delta z)^{1/2} \sum_{n,j} \left| -\phi^*(\omega_n^j, \theta) \right|^{-1/2} \]

\[ \times \tilde{F}(r_T, \omega_n^j - \omega_c) \exp\left[(\Delta z/c)\phi(\omega_n^j, \theta)\right], \] 

(29)

which is valid for \( \theta > 1 \) bounded away from unity as \( \Delta z \to \infty \). The uniform asymptotic description of this first precursor can be found in Refs. 1, 20, 23, and 25. The front of the Sommerfeld precursor \( U_S(r_T, z, t) \) arrives at \( \theta = 1 \) with an infinite instantaneous angular frequency and so travels at the speed of light in vacuum through the dielectric material. As \( \theta \) increases from unity the amplitude of this first precursor rapidly builds to a peak value and thereafter decays as the instantaneous oscillation-frequency chirps downward toward \( \omega_n \) and the attenuation factor increases correspondingly, as is illustrated in Fig. 13(b). The component field \( U_B(r_T, z, t) \) is due to the asymptotic expansion about the near saddle points \( \omega_n^j(\theta) \) of \( \phi(\omega_0, \theta) \) that evolve with \( \theta > 1 \) in the low-frequency region, \( |\omega| \ll \omega_0 \), and describes the Brillouin precursor, whose nonuniform asymptotic description is given by\(^{1,19}\)

\[ U_B(r_T, z, t) \sim (c/2\pi \Delta z)^{1/2} \left| -\phi^*(\omega_n^0, \theta) \right|^{-1/2} \]

\[ \times \tilde{F}(r_T, \omega_n^0 - \omega_c) \exp\left[(\Delta z/c)\phi(\omega_n^0, \theta)\right], \] 

(30a)

for \( 1 < \theta < \theta_1 \), by

\[ U_B(r_T, z, t_1) \sim \frac{\Gamma(1/3)}{2\pi^{3/2} \left| \Delta z/\phi''(\omega_n^0, \theta_1) \right|^{1/3}} \]

\[ \times \exp\left[(\Delta z/c)\phi(\omega_n^0, \theta_1)\right] \] 

at \( \theta = \theta_1 = ct_1/\Delta z \), and by

\[ \text{Eq. (28)} \]
for $\theta > \theta_1$. The uniform\textsuperscript{1,20} and the transitional\textsuperscript{22} asymptotic descriptions provide a continuous description of the Brillouin precursor evolution across the critical space–time point at $\theta = \theta_1$. The Brillouin precursor $U_B(\mathbf{r}_T, z, t)$ is nonoscillatory over the initial space–time domain, $1 < \theta \leq \theta_1$, during which its amplitude builds to a maximum value that occurs near the space–time point $\theta = \theta_0 = n(0)$, where there is no exponential attenuation.\textsuperscript{1,19–22} The space–time point $\theta_1$ is defined\textsuperscript{1,19} as the value that occurs when the two near-first-order saddle points coalesce into a single second-order saddle point, where $\theta_1$ just follows $\theta_0$ for a Lorentz model dielectric. As $\theta$ increases to greater than $\theta_1$, the Brillouin precursor becomes oscillatory with an instantaneous oscillation frequency that chirps upward toward $\omega_0$ and a decreasing amplitude as the attenuation factor increases correspondingly, as is illustrated in the Fig. 14(b). The peak in the Brillouin precursor then travels with the velocity $\nu = c/\theta_0 = c/n(0)$ through the dielectric material. The middle precursor field $A_m(z, t)$ is due to the asymptotic contribution from any additional saddle points that might appear in the intermediate frequency domain, $\omega_0 < |\omega| \leq \omega_m$; a condition for the appearance of such a middle precursor is given in Ref. 21. The final contribution $U_c(\mathbf{r}_T, z, t)$ is due to any poles that are crossed in deforming the original contour $C$ to an Olver-type path $P(\theta)$ through the saddle points.\textsuperscript{1} This contribution to the asymptotic behavior of the propagated field describes the steady-state behavior (if there is one) of the propagated signal that oscillates at the input carrier frequency $\omega = \omega_c$. Important examples for which there is no pole

$$
U_B(\mathbf{r}_T, z, t) \sim (c/2\pi\Delta z)^{1/2} \sum_{j=\pm} \left[-\phi''(\omega_n^j, \theta)\right]^{-1/2} \times \tilde{F}(\mathbf{r}_T, \omega_n^j - \omega_c) \exp\left[(\Delta z/c)\phi(\omega_n^j, \theta)\right]
$$

(30c)

as $\Delta z \rightarrow \infty$, or in a somewhat more complicated form that is given by the linear superposition of fields that are themselves expressed in the form given in Eq. (31). There is now no Sommerfeld precursor, and the field arrives with the evolution of the Brillouin precursor $U_B(\mathbf{r}_T, z, t)$, which is nonoscillatory for all $\theta > \theta_n$, where $\theta_n = [\epsilon(\infty)]^{1/2} > 1$. The asymptotic contribution that is due to the lower-near saddle point results in a Brillouin precursor field that is characteristic of a Debye-type dielectric\textsuperscript{12,13,18} in that it is quasi static over the entire space–time domain $\theta > \theta_1$ and decays with only the propagation distance as $(\Delta z)^{-1/2}$ at the space–time point $\theta = \theta_0$, where $\theta_1$ precedes $\theta_0$ for a Debye model dielectric. The peak of the Brillouin precursor, which occurs at the space–time point $\theta = \theta_0 = n(0)$, where there is no exponential attenuation and the field amplitude decays with the propagation distance only as $(\Delta z)^{-1/2}$, travels at the velocity $\nu = c/\theta_0 = c/n(0)$ through the dielectric material.\textsuperscript{12,13,18}

Note that a real material can be described with a combination of both models but that only the low-frequency Rodard–Powles model can be relevant if the input pulse spectrum is negligible for frequencies that approach the lowest characteristic resonance frequencies of the material.\textsuperscript{13} On the other hand, if the input pulse spectrum spans both the high-frequency resonance and the

![Fig. 13](image1.png)

Fig. 13. (a) Fresnel number evolution for (b) the Sommerfeld precursor field evolution plotted as functions of $\theta = ct/z$ at a fixed propagation distance of $z = 10x_0 = 0.1153$ m with $\omega_c = 5.00 \times 10^{14}$ rad/s in a single-resonance Lorentz model dielectric with $\omega_0 = 9.14 \times 10^{14}$ rad/s, $b_0 = 5.00 \times 10^{13}$ rad/s, and $\delta_0 = 1.43 \times 10^{13}$ rad/s.

![Fig. 14](image2.png)

Fig. 14. (a) Fresnel number evolution for (b) the Brillouin precursor field evolution plotted as functions of $\theta = ct/z$ at a fixed propagation distance of $z = 10x_0 = 0.1153$ m with $\omega_c = 5.00 \times 10^{14}$ rad/s in a single-resonance Lorentz model dielectric with $\omega_0 = 9.14 \times 10^{14}$ rad/s, $b_0 = 5.00 \times 10^{13}$ rad/s, and $\delta_0 = 1.43 \times 10^{13}$ rad/s.
low-frequency relaxation structures, the high-frequency resonance phenomena may dominate the material dispersion to such an extent that the low-frequency relaxation structure is entirely negligible, as occurs for the composite Rocard–Powles–Lorentz model of triply distilled water.\textsuperscript{12}

For either an ultrawideband signal or an ultrashort pulse the beam field breaks up into a sequence of subpulses, as described by either Eq. (28) for a Lorentz-type dielectric or Eq. (31) for a Debye-type dielectric (including the Rocard–Powles extension of this model). Each subpulse $U_j(t, z, t_j)$, $j = S, B, m, c$, is described by its own characteristic frequency evolution. The transverse beam width of each individual subpulse is determined, in part, by this characteristic frequency evolution so that each subpulse has different diffraction characteristics. A convenient general measure of the transverse-beam-width evolution for each subpulse is provided by the inverse of the Fresnel number evaluated at the instantaneous oscillation frequency of the subpulse, where

$$N_j(\theta) = \left(\frac{a^2}{2\pi a} \beta(\omega_{sp}(\theta))\right)^2,$$

for $j = S, B, m, c$. Here $a$ is a measure of the initial beam width at $\Delta z = 0$.

The dynamical evolution of the Sommerfeld and the Brillouin precursor fields and the corresponding Fresnel number evaluations are presented in Figs. 13 and 14 for a single-resonance Lorentz model dielectric with a complex index of refraction:

$$n(\omega) = \left[1 - \frac{b_0^2}{(\omega^2 - \omega_0^2) + 2i\delta_0(\omega)}\right]^{1/2}.$$

In Eq. (33) $\omega_0 = 9.14 \times 10^{14}$ rad/s is the undamped resonance frequency of the dielectric, $\delta_0 = 1.43 \times 10^{13}$ rad/s is the phenomenological damping constant, and $b_0 = 5.0 \times 10^{13}$ rad/s is the plasma frequency. The precursor fields depicted in Figs. 13 and 14 were computed by use of numerically determined saddle-point locations in the uniform asymptotic description for the transient field produced by a Heaviside step-function-modulated signal\textsuperscript{19,21} with a below-resonance carrier frequency of $\omega_c = 5.0 \times 10^{14}$ rad/s at the propagation distance of $\Delta z = 10z_d = 0.1153$ m, where $z_d = \alpha^{-1}(\omega_c)$ is the absorption depth at the carrier frequency. The corresponding Fresnel numbers $N_c(\theta)$ and $N_p(\theta)$, illustrated in Figs. 13 and 14, respectively, were computed for an initial 2-cm beam diameter. The Fresnel number variation depicted in Fig. 13 shows that the Sommerfeld precursor remains geometrically dominated throughout its evolution. This high-frequency subpulse then propagates through the dispersive, attenuative medium with negligible diffractive spreading. The Fresnel number variation depicted in Fig. 14 shows that the quasi-static leading edge ($1 < \theta < \theta_0$) of the Brillouin precursor is diffraction dominated but that, as $\theta$ increases above $\theta_0$, the Fresnel number rapidly increases, and the diffractive spreading correspondingly decreases. Note that the oscillation frequency of the Brillouin precursor is nonzero but purely imaginary for $1 < \theta < \theta_0$. The leading edge of this low-frequency subpulse then propagates through the dispersive, attenuative medium with negligible loss but with maximum diffractive spreading. The remainder of this subpulse suffers increasing loss with decreasing diffractive spreading.

5. DISCUSSION

The results presented in this paper provide a general framework within which a detailed analysis of the coupled spatiotemporal dynamics of several critical canonical problems that arise in ultrawideband-pulsed electromagnetic beam fields can be obtained for nontruncated beam fields. These include, for example, ultrashort-pulsed Hermite–Gaussian and Laguerre–Gaussian beam fields. For a given truncated beam field, asymptotic contributions from the edge-diffracted pulses, as described by Solhaug et al.,\textsuperscript{30} must be included in the formalism. These edge-diffracted pulses can result in a significant alteration in the dynamical field evolution in the focal region. The important result that the accuracy of the paraxial approximation of the propagation kernel increases as either the loss or the propagation distance increases in an attenuative medium is ideally suited for asymptotic methods of analysis for the temporal evolution of the pulsed-beam field because the accuracy of that approximation increases in the well-defined sense of Poincaré\textsuperscript{16} as the propagation distance $\Delta z$ increases to greater than (typically) a single absorption depth $z_d = \alpha^{-1}(\omega_c)$ in the medium at some characteristic frequency $\omega_c$ of the input ultrawideband pulse.

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REFERENCES AND NOTES

7. J. A. Stratton, Electromagnetic Theory (McGraw-Hill, New York, 1941), Sec. 5.18.
8. Both cgs and mks units are employed in this paper through the use of a conversion factor that appears in the double brackets $\llbracket \cdot \rrbracket$ in each affected equation. If this factor is included in the equation it is then in cgs units, provided that
whereas if this factor is omitted the equation is in mks units. If no such factor appears the equation is correct in both systems of units.


