Numerical analysis of the precursor fields in linear dispersive pulse propagation

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A fast, accurate numerical algorithm based on the Laplace transform is presented. By using this algorithm, the high-frequency structure present at the onset of the precursor field for both the delta-function pulse and the unit-step-function-modulated signal has been resolved. A comparison of the results with those obtained by modern asymptotic techniques demonstrates the ability of this numerical method to resolve the high-frequency structure associated with the onset of the Sommerfeld precursor field.

INTRODUCTION

The propagation of a given physical waveform through a linear causally dispersive and absorptive medium is a classic problem that remains a topic of fundamental interest. Much of this interest has focused on finite-duration wave fields of finite total energy for which the dynamic evolution of the pulse signal in space–time must be determined. This problem is of fundamental importance in many physical systems. Examples include acoustic waves in superfluid helium,\textsuperscript{1} radio-wave propagation in the ionosphere,\textsuperscript{2,3} electromagnetic pulse propagation in plasma media,\textsuperscript{4} and optical pulse propagation in both dielectric media\textsuperscript{5,6} and dielectric fiber waveguides.\textsuperscript{7,8}

The seminal analysis of the precursor fields associated with dispersive pulse propagation in a linear dielectric medium was provided by Sommerfeld\textsuperscript{9} and Brillouin\textsuperscript{10,11} in 1914, using the asymptotic method of steepest descents. The recent analysis of Oughstun and Sherman,\textsuperscript{6} using modern asymptotic techniques, has resulted in significant quantitative improvements in the entire description of the propagated field as well as a more precise description of the signal velocity in dispersive media. This analysis has led to a simple mathematical procedure for determining the resultant pulse dynamics in a given dispersive medium (the Lorentz medium) and also provides a clear physical interpretation.\textsuperscript{5,12}

A proof that this new physical interpretation holds for general causally dispersive systems remains to be given and perhaps may be approached only on a case-by-case basis. However, because of the complexity of the analysis of the complex phase behavior necessary for the asymptotic theory, such a case-by-case approach may be prohibitive in all but the most important cases. A more timely approach is to use some appropriate numerical technique that will provide a known degree of accuracy for any given dispersive system. The most straightforward numerical approach is to use the fast-Fourier-transform algorithm in a direct numerical synthesis of the Fourier integral representation of the dispersive pulse propagation problem.\textsuperscript{13} Unfortunately, the computational efficiency of this numerical procedure decreases rapidly (owing to limited memory capacity) when an attempt is made to calculate the fine high-frequency field structure that is characteristic of the onset of the Sommerfeld precursor field. A numerical implementation of the asymptotic theory has also been described\textsuperscript{14} that allows one to calculate directly any desired structure in the propagated field with a known degree of accuracy. Although physically appealing, this technique is numerically intensive and requires a large storage capacity.

In this paper a Laplace-transform method of numerically evaluating the precursor fields associated with dispersive pulse propagation, originally described by Hosono,\textsuperscript{15} is presented. The purpose of this paper is first to provide a careful, correct derivation of this numerical procedure and to state the approximations and limits of the validity of this method. The procedure is then applied in a numerical calculation of the transient phenomena associated with dispersive pulse propagation in a Lorentz medium. This important medium was chosen so that a direct comparison between the results presented here and the recent uniform asymptotic description of the precursor fields\textsuperscript{16} could be obtained. The numerical procedure described here is of special importance because it allows for accurate calculations of the field behavior at the onset of the Sommerfeld precursor without the need for a large amount of memory space or exceedingly time-intensive computations. With this numerical technique, it is then possible to investigate carefully any space–time region of the transient field structure associated with dispersive pulse propagation for any given dispersion relation of appropriate form without the need for performing a complicated asymptotic expansion. The technique also provides a useful comparison for the asymptotic theory in those important cases that warrant such a detailed analysis.

LAPLACE-TRANSFORM REPRESENTATION

Consider a plane electromagnetic wave propagating in the positive z direction through a linear, homogeneous, isotropic, temporally dispersive medium. Let $f(z, t)$ be the ampli-
The spectral amplitude of the signal in the Fourier representation is given by
\[ F(z, i\omega) = \int_{-\infty}^{+\infty} f(z, t) e^{i\omega t} dt, \]  
and the inverse transformation is
\[ f(z, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(z, i\omega) e^{-i\omega t} d\omega, \]
in which the bar through the integral sign denotes the principal part. If \(-i\omega\) is replaced by \(s = \sigma - i\omega\), the spectral amplitude becomes
\[ F(z, s) = \int_{-\infty}^{+\infty} f(z, t) e^{-\sigma t} e^{i\omega t} dt, \]
which is the Fourier transform of \(f(z, t)e^{-\sigma t}\). The inverse transformation is then given by
\[ f(z, t) e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(z, s) e^{i\omega t} d\omega, \]
If it is assumed that \(\sigma\) is constant, then \(d\omega = ids\), and consequently
\[ f(z, t) = \frac{-i}{2\pi} \int_{-\infty}^{+\infty} F(z, s) e^{i\omega t} ds. \]

The signals of interest are such that \(f(z, t) = 0\) for \(t < 0\), so that the above equations can be expressed in terms of the Laplace-transform pair relationship
\[ F(z, s) = \int_{0}^{+\infty} f(z, t) e^{-\sigma t} e^{i\omega t} dt, \]  
\[ f(z, t) = \frac{-i}{2\pi} \int_{-\infty}^{+\infty} F(z, s) e^{i\omega t} ds, \]
where \(Br\) denotes the appropriate Bromwich contour. For a signal propagating in the linear, dispersive medium, the spectral amplitude satisfies the scalar Helmholtz equation
\[ (\nabla^2 + k^2(\omega))F(z, i\omega) = 0, \]  
the general solution of which is given by
\[ F(z, i\omega) = F_+(i\omega) \exp[ik(\omega)z] + F_-(i\omega) \exp[-ik(\omega)z]. \]
Since the wave field is assumed to be propagating in the positive \(z\) direction, \(F_-(i\omega) = 0\) and \(F(z, i\omega) = F_+(i\omega) \exp[ik(\omega)z]\), so that \(F(z, s) = F_+(s) \exp[ik(is)z]\), which gives
\[ f(z, t) = \frac{-i}{2\pi} \int_{Br} F_+(s) \exp[ik(is)z] e^{\sigma t} ds. \]

At \(z = 0\) the boundary value is
\[ f(0, t) = \frac{-i}{2\pi} \int_{Br} F_+(s) e^{\sigma t} ds, \]
and the transform is \(F_+(s) = F(0, s)\), so that one finally obtains
\[ f(z, t) = \frac{-i}{2\pi} \int_{Br} F(0, s) \exp[ik(is)z] e^{\sigma t} ds, \]
where \(F(0, s)\) is the initial spectrum of the signal at the plane \(z = 0\).
\[ f_{\text{app}}(z, t) = \frac{e^{st}}{2} \sum_{n=-\infty}^{+\infty} \frac{1}{2ni} \left[ (-1)^{n} F(0, s) \sum_{m=-\infty}^{\infty} \exp[i k (is) z] ds \right] \]

\[ = \frac{e^{st}}{2t} \sum_{n=-\infty}^{+\infty} (-1)^{n} F \left[ 0, \frac{a + i(n - \frac{1}{2}) \pi}{t} \right] \times \exp \left[ ik \left( \frac{1}{t} s n \right) z \right] \]

\[ = \frac{e^{st}}{2t} \sum_{n=-\infty}^{+\infty} (-1)^{n} \left( F \left[ z, \frac{a + i(n - \frac{1}{2}) \pi}{t} \right] - F \left[ z, \frac{a - i(n - \frac{1}{2}) \pi}{t} \right] \right). \]

(20)

Since \( f(z, t) \) is real valued, \( F(z, s^*) = F^*(z, s) \), and Eq. (20) becomes

\[ f_{\text{app}}(z, t) = \frac{e^{st}}{t} \sum_{n=1}^{+\infty} (-1)^{n+1} \left( F \left[ z, \frac{a + i(n - \frac{1}{2}) \pi}{t} \right] - F \left[ z, \frac{a - i(n - \frac{1}{2}) \pi}{t} \right] \right). \]

(21)

Bromwich's method of transformation\(^{18}\) for slowly convergent alternating series can now be applied to Eq. (21). The first step consists of breaking up \( f_{\text{app}}(z, t) \) into two pieces:

\[ f_{\text{app}}(z, t) = \frac{e^{st}}{t} \left( \sum_{n=1}^{m} (-1)^{n+1} \sum_{n=1}^{+\infty} F_n + \sum_{n=m+1}^{+\infty} D^n F_k \right) \]

(23)

in which \( D^n \) is the Euler difference operator that is defined by the equations

\[ D^n v_n = v_n - v_{n+1}, \]

\[ D^n v_n = D v_n - D v_{n+1} = v_n - 2v_{n+1} + v_{n+2}. \]

In practical cases the remainder is truncated at \( n = m \), and the summation is rewritten as

\[ \sum_{n=m}^{+\infty} D^n F_k \frac{1}{2^n + 1} = \frac{1}{2^{m+1} + 1} \sum_{n=m}^{+\infty} A_{m,n} F_k + \sum_{n=0}^{m} A_{m,n} F_k + \sum_{n=0}^{m} A_{m,n} F_k + \sum_{n=m+1}^{+\infty} \]

(24)

in which the coefficients \( A_{m,n} \) are defined recursively by the expressions

\[ A_{m,m} = 1, \]

\[ A_{m,n-1} = A_{m,n} + \left( \frac{m + 1}{n} \right). \]

as shown in Appendix A. In terms of these new coefficients Eq. (23) becomes

\[ f_{\text{app}}(z, t) = \frac{e^{st}}{t} \left( \sum_{n=1}^{k-1} F_n + \frac{1}{2^{m+1} + 1} \sum_{n=0}^{m} A_{m,n} F_k + \sum_{n=m+1}^{+\infty} \right), \]

(25)

which is the desired result for the numerical evaluation of the propagated field.

Of particular interest is the behavior of a pulse propagating in a linear dispersive dielectric medium. The complex propagation constant \( k(\omega) \) for a single-resonance Lorentz medium is given by

\[ k(\omega) = \frac{\omega}{c} n(\omega) = \frac{\omega}{c} \left( 1 - \frac{\omega_p^2}{\omega^2 + 2i \delta \omega - \omega_0^2} \right)^{1/2}, \]

(26)

in which \( n(\omega) \) is the complex index of refraction of the dielectric medium. The quantity \( k(is) \) appearing in the integral formula [Eq. (12)] is then given by

\[ k(is) = \frac{is}{c} \left( 1 + \frac{\omega_p^2}{s^2 + 2i \delta s + \omega_o^2} \right)^{1/2}, \]

(27)

and the propagated spectrum is

\[ F(z, s) = F(0, s) \exp \left[ -\frac{s z}{c} \left( 1 + \frac{\omega_p^2}{s^2 + 2i \delta s + \omega_0^2} \right)^{1/2} \right]. \]

(28)

**VALIDITY OF THE APPROXIMATION**

The critical approximation in the numerical method is contained in relation (13). For this approximation to be valid, the condition

\[ |\exp[-2(a - st)]| = |\exp[-2(a - \text{Re}(s)t)]| \ll 1 \]

must be satisfied, which in turn requires that \( a \gg \text{Re}(s)t \).

When this condition is met, Eq. (14) can be expanded in an infinite series of the form

\[ E(a, st) = e^{st} \sum_{n=0}^{+\infty} (-1)^{n+1} \exp[-2(a - st)n] \]

\[ = e^{st}[1 - \exp[-2(a - st)] + \exp[-4(a - st)] + \ldots]. \]

(30)

The error introduced by this approximation is simply given by

\[ \delta_{\text{app}} = |E(a, st) - e^{st}| \]

\[ = e^{-2a} \frac{e^{3st}}{1 + \exp[-2(a - st)]} \approx e^{-2a}. \]

(31)

From relation (31) it is obvious that the value of parameter \( a \) must be chosen large enough to ensure that the approximation does not introduce any significant errors.

The other approximation in the algorithm is contained in the truncation of the infinite series in Eq. (23). The magnitude of the term that is left out by this truncation is given by
Influence of the Exponential Parameter $a$
As indicated by relation (31), one must choose the value of the exponential parameter $a$ to be much larger than $\text{Re}(s)$ to obtain accurate results. The ability of the method to perform the reverse transformation is not uniform in the time and frequency domains. At large $t$, the algorithm is restricted to low frequencies, whereas at high frequencies, $t$ must be small. The primary utility of this method in optical problems is not at large $t$. Analytical methods can be used to predict the dynamics of the pulse after it has traveled an extended distance into the medium. However, by using this new numerical Laplace-transform method, it is possible to resolve the fine high-frequency structure that exists at the front of the pulse. Simulations performed for a rectangular pulse (without dispersion) showed that setting $a = 2$ (this was considered a high value for $a$ by Hosono) led to an erroneous description of the front of the pulse. The use of higher values of $a$ corrected this problem. An appropriate value was found to be $a = 15$.

Influence of the Initial Sum Index $k$
It has been shown that the accuracy of Bromwich’s method is greatly enhanced when a large number of terms are summed before the truncation approximation is applied. In this case a large value of $k$ allows more of the higher-frequency components to be taken into account, thereby producing more-complete information. There is virtually no limit to the value of $k$ other than those set by the computer’s precision. The choice of an optimum value for $k$ depends on both the limitations of the machine and the spectrum of the signal studied. Signals composed of only low-frequency components will accept small values, whereas larger values must be chosen if the spectrum contains significant high-frequency components. An important limiting case is provided by the delta-function signal, for which the maximum permissible value of $k$ must be chosen to resolve the signal structure at the front of the propagating pulse.

NUMERICAL STUDY OF THE PRECURSOR FIELDS IN LINEAR DISPERITIVE PULSE PROPAGATION
A problem of fundamental interest to which this numerical algorithm has direct application is that of pulse propagation in a linear dispersive and absorptive medium. The simplest causal dispersive medium of central importance to optics is the single-resonance Lorentz medium whose dispersion relation is given in Eq. (26). Previous research on linear dispersive optical pulse propagation focused on this particular medium for both a unit-step-function-modulated signal and an input delta-function pulse, using both asymptotic analysis and several numerical techniques. The most widespread numerical technique has its origin in the Fourier integral representation of the propagated field and relies on the fast-Fourier-transform algorithm. This numerical approach gives an accurate description of the propagated field behavior, except in the very-high-frequency domain, characteristic of the onset of the Sommerfeld precursor field, in which it is limited by computer storage requirements. In addition, the validity of modern asymptotic techniques becomes questionable at the onset of the Sommerfeld precursor field when the initial pulse spectrum is not sufficiently well behaved at $|\omega| = \omega_c$ since at that point the dominant saddle points are located at infinity. Such is the case for the delta-function pulse whose initial spectrum is unity for all $\omega$.

Until recently there was no practical, independent numerical technique that could be used to provide an accurate independent check of the high-frequency structure associated with dispersive pulse propagation as predicted by asymptotic theory. The Laplace-transform method, proposed originally by Hosono and developed correctly here, has proved to be a useful technique for overcoming these limitations. The algorithm does not set any memory space requirements, so that very-high-frequency structures can be readily modeled simply by taking into account a larger spectral domain at the expense of only longer machine computation time.

Attention is now turned to a numerical study of the precursor fields in a single-resonance Lorentz medium for the input delta-function and unit-step-function fields. For the purpose of comparison with previous uniform asymptotic results, Brillouin’s choices for the medium parameters are used throughout this study. They are given by

$$\omega_0 = 4.0 \times 10^{16}/\text{sec},$$
$$b^2 = 20.0 \times 10^{32}/\text{sec}^2,$$
$$\delta = 0.28 \times 10^{16}/\text{sec}$$

and correspond to a highly absorptive and dispersive medium.
At \( \theta = 1 \) the numerical calculations predict a large spike in the field amplitude, as illustrated in Fig. 3(a), which can be explained best as the remainder of the initial delta-function pulse. This spike is not predicted by the asymptotic theory, and it decreases in amplitude as the propagation distance \( z \) increases. Additional numerical calculations, as presented in Fig. 3(b), show that increasing the parameter \( k \) results in a

**Delta-Function Pulse**

For an input delta-function pulse, \( f(0, t) = \delta(t) \), a propagation distance of \( z = 1 \times 10^{-6} \) m was chosen, since at this distance both the Sommerfeld and the Brillouin precursor fields have not undergone too much attenuation, and the accuracy of the asymptotic theory will be tested severely when it is compared with the purely numerical calculations of this study. At this propagation distance the overall field behavior, shown in Fig. 1, is found to be in excellent agreement with both the nonuniform and the uniform asymptotic descriptions of the propagated field, which consists only of the Sommerfeld and the Brillouin precursors. Notice that these results are plotted as functions of the dimensionless space-time parameter \( \theta = ct/z \) so that \( \theta = 1 \) corresponds to the space–time point that travels at the vacuum speed of light. For this and the following calculations, the parameter \( k \), which is associated with the size of the frequency domain, was set to the value \( 5 \times 10^4 \), unless otherwise specified (a value of \( 5 \times 10^5 \) is considered high, whereas \( 5 \times 10^3 \) is low). For all calculations presented in the paper, the parameter \( m \) was set at its maximum value of 250.

Of particular interest here is the behavior of the field both at and immediately after the point \( \theta = 1 \) at which the validity of the asymptotic expansion of the impulse response becomes questionable. This behavior is depicted in Fig. 2, in which the numerically determined behavior (indicated by the dotted curves) is compared with that described by the uniform asymptotic description of Ref. 12 (indicated by the solid curves) for several expanded views near \( \theta = 1 \). It is clear from Fig. 2 that the asymptotic description of the Sommerfeld precursor field is in excellent agreement with the numerically depicted behavior for values of \( \theta \) immediately after \( \theta = 1 \) but begins to disagree as \( \theta \) increases away from unity. A close examination of Fig. 2(b) shows the appearance of an increasing phase shift as well as a slight amplitude deviation in the uniform asymptotic description as \( \theta \) increases. This difference is due primarily to the approximations made in the distant saddle-point locations, as evidenced in Figs. 9–11 of Ref. 6. Despite this difference, the overall structure of the Sommerfeld precursor field is depicted well by the uniform asymptotic theory of Ref. 16 over the entire range of \( \theta \) values of interest (for \( \theta > 1 \)), even at such a small propagation distance.
Fig. 3. Remnant of the input delta-function pulse at a propagation distance of \( z = 10^{-6} \) m. The precursor field evolution for \( \theta > 1 \) is unobservable in the vertical scale of (a), in which the dotted and the solid curves represent numerical results and asymptotic-expansion results, respectively. The dependence of this field structure on the initial sum parameter \( k \), which is a measure of the spectral domain modeled, is depicted in (b).

Fig. 4. Initial evolution of the Sommerfeld precursor for an input delta-function pulse at a propagation distance of \( z = 10^{-6} \) m for several values of the initial sum index \( k \). The dotted curve represents results for \( k = 8 \times 10^5 \), and the dashed curve represents results for \( k = 1 \times 10^5 \); the solid curve represents the results of the uniform asymptotic theory.

Fig. 5. Expanded views of the initial evolution of the Sommerfeld precursor for an input step-function-modulated signal of carrier frequency \( \omega_c = 10^{16} \) sec at a propagation distance of \( z = 10^{-6} \) m for the following ranges of \( \theta \): (a) 1.000–1.300, (b) 1.000–1.050, and (c) 1.000–1.010. The dotted curves depict the numerically determined behavior, and the solid curves represent the results of the uniform asymptotic theory.
narrowing of the spike's width, accompanied by an increase in its amplitude. This implies that this component of the field is characterized by an infinite frequency of oscillation and consequently is subject to zero dispersion. It therefore lies in what has been called the immature dispersion region, to which the asymptotic theory is inapplicable (see the introduction to Ref. 6). The size of the spike increases linearly with the parameter \( k \). As more of the high-frequency components are taken into account, the structure obtained becomes closer to a delta function. However, \( k \) cannot be increased indefinitely because of machine limitations. As the precision limit of the computer is approached, the added terms contain more error, and the resulting signal structure is no longer correct. The observation that the oscillations exhibited in Fig. 3(b) are not centered around zero for \( \Theta > 1 \) is an indication of the presence of the Sommerfeld precursor and of the superposition of this precursor upon the remnant of the delta function. An infinite spectral domain cannot be taken into account, and the truncation leaves out components that, if present, would cancel the field oscillations before \( \Theta = 1 \) and the similar structure after \( \Theta = 1 \).

Finally, notice from Fig. 2(c) that the numerically determined field behavior does not vanish identically for \( \Theta < 1 \) as is demanded by theory. This structure is just an artifact of the numerical algorithm. As shown in Fig. 4, an increase in the value of the parameter \( k \) decreases the amplitude of the field structure for \( \Theta < 1 \). This clearly shows that the oscillations for \( \Theta < 1 \) are unphysical and could nearly be eliminated by proceeding to a sufficiently large value of \( k \), provided that the computer used has a matching precision.

**Unit-Step-Function-Modulated Signal**

For an input unit-step-function-modulated signal, \( f(0, t) = u(t)\sin(\omega_c t) \), in which \( u(t) \) is the Heaviside unit step function, which is 0 for \( t < 0 \) and 1 for \( t > 0 \). For the calculations presented here, the angular frequency of the input signal was set at \( \omega_c = 1 \times 10^{16} \text{sec}^{-1} \), and a propagation distance of \( z = 1 \times 10^{-6} \text{m} \) was chosen. The overall field behavior is found to be in excellent agreement with both the nonuniform and the uniform asymptotic descriptions of the propagated field, which consist of first the Sommerfeld and then the Brillouin precursor fields, followed by the main signal evolution (this overall field structure and the numerical determination of the signal velocity are treated in detail in a companion paper)

Of particular interest here is the field behavior during the arrival and evolution of the Sommerfeld precursor field. This is depicted in Fig. 5, in which the numerically determined behavior (indicated by the dotted curves) is compared with that described by the uniform asymptotic expansion of Ref. 12 (indicated by the solid curves) for several expanded views near \( \Theta = 1 \). It is immediately clear from Fig. 5 that the asymptotic description of the Sommerfeld precursor field is in remarkable agreement with the numerically depicted behavior for values of \( \Theta \) at and immediately after \( \Theta = 1 \), as shown in Fig. 5(c), but it begins to disagree as \( \Theta \) increases away from unity. Notice that the disagreement is more pronounced here than for the case of the delta-function pulse. This is due to the fact that, in the present case, the spectrum of the initial pulse envelope is a function of \( \omega \), so that in the asymptotic evaluation of the propagated field the saddle-point approximations have a greater influence on the field behavior than that found in the delta-function pulse.

**Instantaneous Angular Frequency of Oscillation of the Precursor Fields**

An important qualitative measure of the accuracy of the numerical algorithm presented here is the instantaneous
angular frequency of oscillation of the propagated field structure. Explicit (albeit approximate) algebraic expressions for this quantity have been obtained by using the modern asymptotic theory and have been shown to generalize to a frequency $\omega_G$ that may be obtained directly from the energy-transport velocity in the dispersive medium. Previous numerical calculations of the instantaneous angular frequency of oscillation of the Sommerfeld and Brillouin precursor fields, obtained from the numerically determined field evolution by using the fast-Fourier-transform algorithm, have been found to be in good agreement with the frequency $\omega_G$, thereby reinforcing the validity of the new physical model of dispersive pulse dynamics presented in Ref. 5. These calculations are repeated here with a greater degree of accuracy, and they confirm the earlier results.

The instantaneous angular frequency of oscillation is obtained from the numerically determined field evolution of the input delta-function pulse by calculating the half-period between two successive zero crossings and assigning that frequency to the middle of that measurement interval. This is clearly only an approximation of the instantaneous angular frequency, which will yield the average frequency of oscillation over the measurement interval, provided that the actual value is monotonically varying over that interval; the assignment of that frequency to the middle of the measurement interval is purely a matter of choice. The results are depicted as the triangular points in Fig. 6 along with the curves for the instantaneous angular frequency $\omega_S$ for the Sommerfeld precursor and $\omega_B$ for the Brillouin precursor, as obtained from the asymptotic theory. The dashed curve in the figure depicts the frequency $\omega_G$ that is obtained from the energy-transport velocity. The agreement between the numerically determined values and the frequency $\omega_B$ is seen to be exceptionally good. Similar results are obtained for the precursor fields associated with the unit-step-function-modulated signal.

The discontinuous drop in the angular frequency $\omega_B$ to zero at $\theta = \theta_{SB} = 1.334$ arises in the asymptotic theory and denotes the point at which the Brillouin precursor becomes asymptotically dominant over the Sommerfeld precursor. Notice that the numerical data begin to oscillate rapidly at this point because the total field is the superposition of the rapidly oscillating Sommerfeld precursor upon the nonoscillating portion of the Brillouin precursor, as shown in Fig. 1. The Brillouin precursor begins to oscillate at $\theta = \theta_1 = \theta_0$, in which $\theta_0 = n(0)$ is the zero-frequency value of the refractive index of the medium.

CONCLUSIONS

An efficient numerical algorithm for the inversion of Laplace-transform integrals has been developed carefully here and has been applied to the physically important problem of the precursor fields associated with linear dispersive pulse propagation. This numerical technique permits accurate investigation of the extremely high-frequency structure associated with the initial evolution of the Sommerfeld precursor field. The results presented here are the first validation to our knowledge of the asymptotic description of this fundamental field structure. The algorithm presented here should have general applicability to the evaluation of Fourier–Laplace-type integrals that arise in the analysis of transient field problems. The application of this numerical technique to the determination of the signal velocity in dispersive pulse propagation is presented in a companion paper. The same type of calculation, with a given degree of accuracy, may readily be performed for other types of dispersion relation.

APPENDIX A: TRANSFORMATION FROM THE COEFFICIENTS $D^i$ TO THE COEFFICIENTS $A_{mn}$

The remainder in Eq. (23) can be defined as a series $R(k, m)$, where

$$R(k, m) = \sum_{i=0}^{m} D^i F_k \frac{1}{2^{i+1}}.$$  \hspace{1cm} (A1)

The Euler difference operator $D^i$, defined as

$$D^i F_k = F_k,$$
$$D^1 F_k = F_k - F_{k+1},$$
$$D^2 F_k = F_k - 2F_{k+1} + F_{k+2},$$
$$D^3 F_k = \sum_{j=0}^{i} (-1)^{(i-j)} (\frac{i}{j})^2 F_{k+j},$$  \hspace{1cm} (A2)

in which $(\frac{j}{i})$ denotes the binomial coefficients, then leads to the expression

$$R(k, m) = \frac{1}{2^{m+1}} \left[ \sum_{i=0}^{m} \sum_{j=0}^{i} (-1)^{(i-j)} (\frac{i}{j})^2 m^{i-j} F_{k+j} \right].$$  \hspace{1cm} (A3)

The coefficient of $F_{k+j}$ is then given by

$$\sum_{i=0}^{m} \left( \frac{i}{j} \right)^2 m^{i-j} = A_{k+1,m+1}.$$  \hspace{1cm} (A4)

APPENDIX B: COMPUTATION OF THE COEFFICIENTS $A_{mn}$

The recursion relation for $A_{i,j-1}$ (valid for $j < i$) is given by

$$A_{i,j-1} = A_{i,j} + \left( \frac{i-1}{i} \right),$$  \hspace{1cm} (B1)

which can be rewritten as

$$A_{i,j} = \sum_{k=j}^{i} \frac{k}{i}.$$  \hspace{1cm} (B2)

By recurrence one may show that $A_{i,j} = A_{i-1,j-1} + A_{i-1,j}$ as follows. First

$$A_{1,1} = 1, \quad A_{2,2} = 1,$$
$$A_{2,1} = \left( \frac{1}{2} \right) + \left( \frac{2}{2} \right) = A_{1,0} + A_{1,1} = 3,$$  \hspace{1cm} (B3)

whereas for the general term one has
\[ A_{i-1,j+1} + A_{i-1,j} = \sum_{k=j}^{i-1} \left( \begin{array}{c} k \\ j \end{array} \right) \sum_{k=j}^{i-1} \left( \begin{array}{c} k \\ j \end{array} \right) \]

\[ = \sum_{k=j}^{i-1} \left( \begin{array}{c} k \\ j \end{array} \right) + \sum_{k=j}^{i-1} \left( \begin{array}{c} k \\ j \end{array} \right) \]

\[ = \left( \begin{array}{c} i-1 \\ j \end{array} \right) + \sum_{k=j}^{i-1} \left( \begin{array}{c} k-1 \\ j \end{array} \right) + \left( \begin{array}{c} i-1 \\ j \end{array} \right) \]

\[ = \left( \begin{array}{c} i \\ j \end{array} \right) + \sum_{k=j}^{i-1} \left( \begin{array}{c} k \\ j \end{array} \right) = A_{i,j} \]  

(B4)

The following matrix is an example of the ease with which one can compute the \( A_{m,n} \) coefficients. Notice that column zero is used for computational purposes only; it contains the power of 2 associated with the row considered and is an artifice to simplify and to generalize the computation of the coefficients \( A_{m,n} \). Each row (1, 2, 3... ) contains the coefficients used in a given simulation and corresponds to the value of the parameter \( n \). The actual values are stored in the matrix, starting in column 1. Any coefficient is derived from the values in column 0 and in the unity diagonal by using the following scheme: once the coefficients in row \( n \) have been obtained, those located in row \( n + 1 \) are obtained simply by adding the coefficient located in row \( n \) directly above and the one at the left of the latter, as is readily evident in the example below:

| column | row | 0 | 1 | 2 | 3 | 4 | ...
|--------|-----|---|---|---|---|---|---
| 0      | 1   |   |   |   |   |   |
| 1      | 2   | 1 |   |   |   |   |
| 2      | 4   | 3 | 1 |   |   |   |
| 3      | 8   | 7 | 4 | 1 |   |   |
| 4      | 16  | 15| 11| 5|1  |   |
|        | 2^n | . | . | . | . | . | . |

This scheme is simpler to use than the direct method (which uses factorials), and it permits values of \( m \) in the range \([0, 250]\), as opposed to \([0, 50]\), for the required factorials.

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