Uniform asymptotic description of electromagnetic pulse propagation in a linear dispersive medium with absorption (the Lorentz medium)

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The uniform asymptotic description of electromagnetic pulse propagation in a single-resonance Lorentz medium is presented. The modern asymptotic theory used here relies on Olver's saddle-point method [Stud. Appl. Math. Rev. 12, 228 (1970)] together with the uniform asymptotic theory of Handelsman and Bleistein [Arch. Ration. Mech. Anal. 35, 267 (1969)] when two saddle points are at infinity (for the Sommerfeld precursor), the uniform asymptotic theory of Chester et al. [Proc. Cambridge Philos. Soc. 53, 599 (1957)] for two neighboring saddle points (for the Brillouin precursor), and the uniform asymptotic theory of Bleistein [Commun. Pure Appl. Math. 19, 353 (1966)] for a saddle point and nearby pole singularity (for the signal arrival). Together with the recently derived approximations for the dynamical saddle-point evolution, which are accurate over the entire space–time domain of interest, the resultant asymptotic expressions provide a complete, uniformly valid description of the entire dynamic field evolution in the mature dispersion limit. Specific examples of the delta-function pulse and the unit-step-function-modulated signal are considered.

1. INTRODUCTION

The classical theory of electromagnetic pulse propagation in a locally linear, homogeneous, isotropic, causally dispersive medium, as described by the Lorentz model, beginning with the seminal analyses of Sommerfeld1 and Brillouin2,3 and continuing up to the modern asymptotic analysis of Oughstun and Sherman,4,5 has been concerned primarily with the dynamic evolution of the saddle points associated with the complex phase function of the medium. This evolution provides a convenient means for describing the fundamental dynamic structure of the precursor fields as well as providing a precise definition of the signal arrival and signal velocity associated with dispersive pulse propagation. However, the asymptotic description of the propagated field that is presented in these papers is known to degenerate at certain space–time points in the field evolution.4,5 This restriction is removed in the present study, in which a complete, uniformly valid asymptotic description of the entire field evolution is provided. For convenience, the notation used in Ref. 5 is used throughout this paper.

The integral representation of the propagated plane-wave pulse in the half-space \( z > 0 \) is given by

\[
A(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \exp \left[ \frac{2}{c} \phi(\omega, \theta) \right] d\omega, \tag{1.1}
\]

where

\[
f(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \tag{1.2}
\]

is the temporal Fourier spectrum of the initial pulse \( f(t) = A(0, t) \) on the plane \( z = 0 \). If \( f(t) = 0 \) for \( t < 0 \), then the integral expression in Eq. (1.1) is taken to be a Laplace representation in which the contour of integration \( C \) in the complex \( \omega \) plane is the straight line \( \omega = \omega' + ia \), where \( a \) is a fixed positive constant that is greater than the abscissa of absolute convergence6 for the function \( f(t) \) and where \( \omega' = \Re(\omega) \) ranges from negative to positive infinity.

The complex phase function appearing in Eq. (1.1) is given by

\[
\phi(\omega, \theta) = i\omega[n(\omega) - \theta], \tag{1.3}
\]

where \( n(\omega) = [\epsilon(\omega)]^{1/2} \) is the complex index of refraction of the dispersive medium with dielectric permittivity \( \epsilon(\omega) \) and where

\[
\theta = \frac{ct}{z} \tag{1.4}
\]

is a dimensionless parameter that characterizes a space–time point in the field, the constant \( c \) being the speed of light in vacuum. For a dielectric Lorentz medium with a single resonance frequency the complex index of refraction is given by

\[
n(\omega) = \left( 1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\omega} \right)^{1/2}. \tag{1.5}
\]

Here \( b^2 = 4\pi Ne^2/m \) is the square of the plasma frequency of the medium, \( N \) is the number density of electrons of charge \( e \) and mass \( m \) that are harmonically bound with the un-
damped resonance frequency $\omega_0$, and $\delta$ is the associated phenomenological damping constant. The Lorentz model is used here because it is a causal model, with the complex index of refraction [Eq. (1.5)] satisfying the Kramers–Kronig relations, and also because it is of general importance to optics. A complete description of the analytic structure of $n(\omega)$ and $\phi(\omega, \theta)$ in the complex $\omega$ plane may be found in Ref. 5.

The uniform asymptotic description of the dynamic field evolution in the mature dispersion regime is developed here for a general initial pulse-modulated sine wave of applied signal frequency $\omega_c$, where

$$ f(t) = u(t)\sin(\omega_c t) $$

and where $u(t)$ is the real-valued initial envelope function of the input pulse that is zero for $t < 0$. With this substitution the integral representation [Eq. (1.1)] for the propagated field becomes

$$ A(z, t) = \frac{1}{2\pi} \text{Re} \left\{ \int_{\infty - i\infty}^{\infty + i\infty} \hat{u}(\omega - \omega_c) \exp \left( \frac{z}{c} \phi(\omega, \theta) \right) d\omega \right\} $$

for $z \geq 0$, where $\hat{u}(\omega)$ denotes the Laplace transform of $u(t)$. The integral representations given in Eqs. (1.1) and (1.7) form the basis of the problem considered in this paper. Both representations apply when the initial pulse shape possesses a sufficiently large exponential decay as $t$ approaches negative infinity to ensure the existence of the appropriate spectrum. Furthermore, with the use of generalized function theory, these representations also apply to the behavior of strictly monochromatic fields. Of specific interest here are the following two canonical problems.

A. First Canonical Problem: The Delta-Function Pulse and the Impulse Response of the Lorentz Medium

The first canonical problem treats the propagation of an input delta-function pulse at time $t = 0$,

$$ f(t) = \delta(t) $$

whose spectrum is given by

$$ \hat{f}(\omega) = 1 $$

for the representation given by Eq. (1.1); one may also use the representation given by Eq. (1.7) with $\hat{u}(\omega - \omega_c) = -i$. In either case the propagated scalar wave disturbance is given by the integral representation

$$ A(z, t) = \frac{1}{2\pi} \text{Re} \left\{ \int_{\infty - i\infty}^{\infty + i\infty} \exp \left( \frac{z}{c} \phi(\omega, \theta) \right) d\omega \right\} $$

This field yields the impulse response of the model medium.

B. Second Canonical Problem: The Unit-Step-Function-Modulated Signal

For a unit-step-function-modulated signal the initial envelope of the pulse is given by the Heaviside unit step function,

$$ u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} $$

so that the input field abruptly begins to oscillate harmonically in time at $t = 0$ and continues indefinitely with a constant amplitude and frequency. The spectrum of this initial pulse envelope is then given by

$$ \hat{u}(\omega) = \int_0^\infty e^{i\omega t} dt = \frac{i}{\omega} \quad \text{for } \text{Im}(\omega) > 0 $$

and the integral representation of the propagated scalar wave disturbance is

$$ A(z, t) = -\frac{1}{2\pi} \text{Re} \left\{ \int_{-\infty - i\infty}^{-\infty + i\infty} \exp \left[ \frac{z}{c} \phi(\omega, \theta) \right] d\omega \right\} $$

for $t > 0$ and is zero for $t < 0$. This field is precisely the one treated by Sommerfeld and Brillouin in their classical treatments of dispersive wave propagation.

Although the integrals appearing in Eqs. (1.9) and (1.11) may be evaluated to a given degree of accuracy with a variety of numerical techniques, such analysis by itself provides little, if any, physical understanding of dispersive pulse propagation phenomena. A complete understanding can be obtained only through a rigorous analytical treatment. The detailed understanding obtained from both the general asymptotic analysis and its application to the selected canonical problems may then be applied directly to other initial pulse shapes, which may themselves be analyzed conveniently with well-defined numerical procedures.

2. Procedure for the Uniform Asymptotic Description of the Propagated Field

It was proved in Ref. 5 that, if the initial envelope $u(t)$ of the field at the plane $z = 0$ is zero for $t < 0$, then the propagated field $A(z, t)$ in the half-space $z > 0$ is zero for all $\theta = ct/z < 1$, provided that both $u(t)$ and $du/dt$ are bounded for all $t > 0$. Attention may therefore be restricted to the asymptotic field behavior for $\theta \geq 1$.

The uniform asymptotic analysis of the propagated field $A(z, t)$ for $\theta \geq 1$ begins by expressing the integral representation of $A(z, t)$ in terms of an integral $I(z, \theta)$ with the same integrand but with a new contour of integration $P(\theta)$ to which the original contour of integration may be deformed and which is divisible into a sum of subpaths, each of which is an Olver-type path with respect to one of the saddle points of the complex phase function $\phi(\omega, \theta)$, as described in Ref. 5. It is assumed here that the spectral function $\hat{u}(\omega - \omega_c)$ is an analytic function of complex $\omega$, regular in the entire complex $\omega$ plane except at a countable number of isolated points where it may exhibit poles. Any poles of $\hat{u}(\omega - \omega_c)$ that are crossed when the original contour is deformed to $P(\theta)$ are encircled in the process in the clockwise sense so that the integral representation $A(z, t)$ and the integral $I(z, \theta)$ are related by

$$ A(z, t) = I(z, \theta) - \text{Re}[2\pi i \Lambda(\theta)] \quad \text{for } \text{Im}(\omega) > 0 $$

where

$$ \Lambda(\theta) = \sum_p \text{Res} \left\{ \frac{i}{2\pi} \hat{u}(\omega - \omega_c) \exp \left[ \frac{z}{c} \phi(\omega, \theta) \right] \right\} $$

is the sum of the residues of the poles that were crossed and where $I(z, \theta)$ is defined by

$$ I(z, \theta) = \frac{1}{2\pi} \text{Re} \left\{ \int_{P(\theta)} i \hat{u}(\omega - \omega_c) \exp \left[ \frac{z}{c} \phi(\omega, \theta) \right] d\omega \right\} \quad \text{for } \text{Im}(\omega) > 0.
Notice that $A(a)$ changes discontinuously with the parameter $\theta$ as the path $P(\theta)$ crosses the poles of $\mathcal{N}(\omega - \omega_0)$. For a single-resonance Lorentz medium there are two sets of saddle points, a pair of distant saddle points $SP_D^+$ and a pair of near saddle points $SP_N^+$, whose dynamical evolution with $\theta$ was described completely in Ref. 5. In Ref. 5 it was also shown that the distant saddle-point locations are given by

$$\omega_{SP_D'} = \pm \xi(\theta) + \delta i [1 - \eta(\theta)],$$

(2.4)

with

$$\xi(\theta) = \left(\omega_0^2 - \delta^2 + \frac{b^2\delta^2}{\theta^2 - 1}\right)^{1/2},$$

(2.5a)

$$\eta(\theta) = \frac{\delta^2/27 + b^2/(\theta^2 - 1)}{\xi^2(\theta)}.$$  

(2.5b)

At $\theta = 1$ the distant saddle points are at $\pm \infty = -2i\delta$, and as $\theta \to \infty$ they approach the outer branch points

$$\omega_{D} = \pm (\omega_0^2 - \delta^2)^{1/2} - \delta i,$$

(2.6)

of the complex phase function $\phi(\omega, \theta)$, where

$$\omega_1 = (\omega_0^2 + b^2)^{1/2}. $$

(2.7)

The $\theta$ dependences of the complex phase function and its second derivative with respect to $\omega$ at the distant first-order saddle points are then found to be given by

$$\phi(\omega_{SP_D'}, \theta) \equiv -i \left\{(1 + \eta(\theta))(\theta - 1) + \frac{b^2(1 - \eta(\theta))^2}{\xi(\theta) + \delta^2(1 - \eta(\theta))^2}\right\}$$

$$= i\xi(\theta)\left[\theta - 1 + \frac{b^2/2}{\xi(\theta) + \delta^2(1 - \eta(\theta))^2}\right],$$

(2.8a)

$$\phi^{(2)}(\omega_{SP_D'}, \theta) \equiv -i \left\{|\pm \xi(\theta) + \delta i[1 - \eta(\theta)]|^{1/2}\right\}.$$  

(2.8b)

These approximations provide an accurate description of the actual distant-saddle-point behavior for all $\theta \geq 1$ and reduce to the expressions given by Brillouin$^{2,3}$ as $\theta \to 1^+$. The near-saddle-point locations are given by

$$\omega_{SP_N}(\theta) \equiv i[\pm \delta(\theta) - (2/3)\delta(\theta)],$$

(2.9a)

$$\omega_{SP_N}(\theta) \equiv -(2\delta/3\alpha)\delta, \quad \theta = \theta_1,$$

(2.9b)

$$\omega_{SP_N}(\theta) \equiv \pm \delta(\theta) - (2/3)i\delta(\theta), \quad \theta > \theta_1,$$

(2.9c)

with

$$\psi(\theta) = \left[\frac{\omega_0^2(\theta^2 - \theta_0^2)}{\theta^2 - \theta_0^2 + \frac{3\alpha b^2}{\omega_0^2}} - \delta^2 \left(\frac{\theta^2 - \theta_0^2 + \frac{2b^2}{\omega_0^2}}{\theta^2 - \theta_0^2 + \frac{3\alpha b^2}{\omega_0^2}}\right)\right]^{1/2},$$

(2.10a)

$$\xi(\theta) = \frac{3}{2} \frac{\theta^2 - \theta_0^2 + \frac{2b^2}{\omega_0^2}}{\theta^2 - \theta_0^2 + \frac{3\alpha b^2}{\omega_0^2}}.$$  

(2.10b)

where the parameter $\alpha$ is given by

$$\alpha = 1 - \frac{\delta^2}{3\omega_0^2\omega_1^2}(4\omega_1^2 + b^2).$$

(11.1)

Here

$$\theta_0 = n(0) = \left(1 + \frac{b^2}{\omega_0^2}\right)^{1/2},$$

(12.1)

$$\theta_1 = \theta_0 + \frac{2b^2}{\delta\omega_0^4}(3\alpha b^2 - 4\delta^2).$$

(13.1)

For $1 \leq \theta < \theta_1$ the two near first-order saddle points are along the imaginary axis situated symmetrically about the point $-(2\delta/3\alpha)i$, at $\theta = \theta_1$ they have coalesced into a single second-order saddle point, and for $\theta > \theta_1$ they are situated symmetrically about the imaginary axis and approach the inner branch points

$$\omega_{SP} = \pm (\omega_0^2 - \delta^2)^{1/2} - \delta i$$

(14.1)

as $\theta \to \infty$. The $\theta$ dependences of the complex phase function and its next nonvanishing higher-order derivative with respect to $\omega$ at the appropriate near saddle points are then found to be given by

$$\phi(\omega_{SP_N}, \theta) \equiv \frac{1}{3} \left[2\delta\xi(\theta) - 3\delta(\theta)\right](\theta_0 - \theta) + \frac{b^2}{54\delta\omega_0^4}$$

$$\times [2\delta\xi(\theta) - 3\delta(\theta) + 3\alpha\delta(\theta)] + 3\alpha\delta(\theta)],$$

(15.1a)

$$\phi^{(2)}(\omega_{SP_N}, \theta) \equiv -\frac{b^2}{\theta_0^4} \left[2\delta(1 - \alpha\delta(\theta)) + 3\alpha\delta(\theta)\right]$$

(15.1b)

for $1 \leq \theta < \theta_1$, where $\omega_{SP_N} = \omega_{SP_N^+}$ is the upper near saddle point (the lower near saddle point does not contribute over this $\theta$ range);

$$\phi(\omega_{SP_N}, \theta) \equiv \frac{2\delta}{3\alpha} \left[\theta_0 - \theta + \frac{4\delta^2\alpha^2}{9\alpha^2\delta\omega_0^4}\right],$$

(16a)

$$\phi^{(3)}(\omega_{SP_N}, \theta) \equiv 3\delta \frac{\alpha b^2}{\theta_0^4},$$

(16b)

for $\theta = \theta_1$ when the two near first-order saddle points have coalesced into a single second-order saddle point; and

$$\phi(\omega_{SP_N}, \theta) \equiv -i \left\{\frac{2\delta}{3} \left[\delta(\theta) - \theta + \frac{2b^2}{\theta_0^4} \left[1 - \alpha\delta(\theta)\right]\right]^\frac{2}{3} - \frac{3\delta}{2}\right\}$$

$$\times \left[\frac{2\delta^2}{\theta_0^4} + 2\theta_0^2 - 1\right] \right\}$$

(17.1a)

$$\phi^{(2)}(\omega_{SP_N}, \theta) \equiv \frac{b^2}{\theta_0^4} \left[2\delta(\theta_1 - \theta) + 3\alpha\delta(\theta)\right]$$

(17.1b)

for $\theta > \theta_1$. These expressions provide an accurate description of the actual behavior at the near saddle points for all $\theta \geq 1$ and reduce to the expressions given by Brillouin$^{2,3}$ as $\theta \to 1^+$. K. E. Oughstun and G. C. Sherman
The contour $P(\theta)$ can always be chosen so that it passes through the upper near saddle point $SP_1$ and the two distant saddle points $SP_d^\pm$ for $1 \leq \theta \leq \theta_1$ (noting that $SP_1 = SP_2 = SP_N$ at $\theta = \theta_1$) and through all four saddle points for $\theta > \theta_1$, such that the path evolves in a continuous fashion as $\theta$ varies over $\theta \geq 1$ and can be divided into the desired Olver-type subpaths with respect to each appropriate saddle point. An example of such a path $P(\theta)$ and its component subpaths is illustrated in Fig. 1. For $\theta$ in the range $1 \leq \theta \leq \theta_1$ the component subpaths are $P_{D^{-}}(\theta)$, $P_1(\theta)$, and $P_{D^{+}}(\theta)$, and for $\theta > \theta_1$ they are $P_{D^{-}}(\theta)$, $P_{N^{-}}(\theta)$, $P_{N^{+}}(\theta)$, and $P_{D^{+}}(\theta)$. The subpaths $P_{D^{-}}(\theta)$ and $P_{N^{-}}(\theta)$ are Olver-type paths with respect to the saddle points $SP_d^-$ and $SP_N^-$ respectively, and the subpath $P_1(\theta)$ is an Olver-type path with respect to the upper near saddle point $SP_1$.

The integral $I(z, \theta)$ given in Eq. (2.3) may then be expressed as a sum of integrals with the same integrand over the component subpaths, so that

$$I(z, \theta) = I_D^-(z, \theta) + I_L(z, \theta) + I_D^+(z, \theta), \quad 1 \leq \theta \leq \theta_1,$$

$$(2.18a)$$

$$I(z, \theta) = I_D^-(z, \theta) + I_N^-(z, \theta) + I_N^+(z, \theta) + I_D^+(z, \theta), \quad \theta > \theta_1,$$

$$(2.18b)$$

where $I_D^-(z, \theta)$ and $I_N^+(z, \theta)$ denote the contour integrals taken over the paths $P_{D^{-}}(\theta)$ and $P_{N^{+}}(\theta)$ respectively, and where $I_L(z, \theta)$ denotes the contour integral taken over the path $P_1(\theta)$. In order to obtain a uniform asymptotic approximation of the integral representation of the field $A(z, \theta)$, one must obtain asymptotic approximations of the integrals appearing on the right-hand sides of Eqs. (2.18) that are uniformly valid in $\theta$ over their respective $\theta$ domains of interest. A description of this procedure is given here.

If the two distant saddle points $SP_d^\pm$ do not pass too near any poles of the spectral function $\hat{u}(\omega - \omega_c)$, then the results of Handel'sman and Bleistein,12 described in Appendix A, can be applied to obtain a uniform asymptotic approximation of the quantity $[I_D^{-}(z, \omega) + I_D^{+}(z, \omega)]$ of the form

$$I_D^-(z, \omega) + I_D^+(z, \omega) = A_d(z, \omega) + R(z, \omega),$$

$$(2.19)$$

where $A_d(z, \omega)$ is obtained from Eq. (A6) and an estimate of the remainder as $z \to \infty$ is given in Eq. (A7) of Appendix A below. Equation (2.19) is uniformly valid for all $\theta > 1$ so long as both of the distant saddle points remain isolated from any poles of $\hat{u}(\omega - \omega_c)$. For values of $\theta$ bounded away from unity from above, this expression reduces to the result obtained by applying Olver's method11 directly to both $I_D^-(z, \omega)$ and $I_D^+(z, \omega)$ and adding the results.

If the near saddle points $SP_1$ and $SP_N^+$ do not pass too close to any poles of the spectral function $\hat{u}(\omega - \omega_c)$, then the results of Chester et al.,13 described in Appendix B, can be applied to obtain uniform asymptotic approximations of $I_1(z, \theta)$ and $I_N^{+}(z, \theta)$ in the form

$$I_1(z, \theta) = A_1(z, \omega) + R'(z, \omega), \quad 1 \leq \theta \leq \theta_1,$$

$$(2.20a)$$

$$I_N^-(z, \omega) + I_N^+(z, \omega) = A_d(z, \omega) + R'(z, \omega), \quad \theta > \theta_1,$$

$$(2.20b)$$

where $A_d(z, \omega)$ and an estimate of the remainder $R'(z, \omega)$ as $z \to \infty$ are given in Eq. (B4) below. Taken together, Eqs. (2.20a) and (2.20b) yield an asymptotic approximation of $[I_D^-(z, \omega) - I_D^+(z, \omega)]$ that is uniformly valid for all $\theta \geq 1$ so long as the near saddle points remain isolated from the poles of $\hat{u}(\omega - \omega_c)$. For values of $\theta$ bounded away from $\theta_1$ from below, Eq. (2.20a) reduces to the result that would be obtained by applying Olver's method directly to obtain the asymptotic approximation of $I_1(z, \theta)$. Similarly, for values of $\theta$ bounded away from $\theta_1$ from above, Eq. (2.20b) reduces to the result that would be obtained by applying Olver's method directly to obtain the asymptotic approximations of $I_N^{-}(z, \omega)$ and $I_N^{+}(z, \omega)$ and summing the results.

Consider now the case in which either one of the distant saddle points $SP_d^\pm$ approaches (as $\theta$ varies) a pole of $\hat{u}(\omega - \omega_c)$ that is located in a region of the complex $\omega$ plane bounded away from the limiting value $\omega = -2\theta_1$ approached by $SP_d^\pm$ as $\theta \to 1^+$. Then the methods described in Appendixes A and C, which combine the results of Olver11 with the uniform expansion of Bleistein,14,15 can be applied to obtain an asymptotic approximation of $[I_D^{-}(z, \omega) + I_D^{+}(z, \omega)]$ that is of the form

$$\text{Fig. 1. Deformed contour of integration $P(\theta)$ through the relevant saddle points of $\hat{u}(\omega, \theta)$. The dashed curves indicate the isotimic contours of $X(\omega, \theta)$ = Re[$\omega(\omega, \theta)$] through the saddle points, and the shaded areas indicate the regions of the complex $\omega$ plane wherein $X(\omega, \theta)$ is less than that at the relevant saddle point.}$$
with $A_z(z, t)$ given by the same expression as in Eq. (2.19). Since the pole is bounded away from the limiting values of \( \omega_{SPD}(\theta) \) as \( \theta \to 1^+ \), the saddle point and pole do not interact for values of \( \theta \) near unity. Hence the asymptotic approximation of \( I_D^-(z, \theta) + I_D^+(z, \theta) \) is determined by applying the method of Appendix A. \( C_B^c(z, t) \) is asymptotically negligible, and Eq. (2.21) reduces to Eq. (2.19) for values of \( \theta \) near 1. For values of \( \theta \) bounded away from unity from above, the results of Appendix C are applicable, and Eq. (2.21) is obtained from Eq. (C5a), with \( A_z(z, t) \) being given by the first term and \( C_B^p(z, t) \) being given by the second term. The resulting expression for \( A_z(z, t) \) is the same as that in Eq. (2.19). Hence \( A_z(z, t) \) in Eq. (2.21) is given by the same expression for \( A_z(z, t) \) in Eq. (2.19) for all \( \theta \geq 1 \). The quantity \( C_B^p(z, t) \) is asymptotically negligible if either of the distant saddle points \( SP_{D^p} \) does not approach a pole of the spectral function \( u(\omega - \omega_d) \), in which case Eq. (2.21) reduces to Eq. (2.19).

Consider next the case where either of the near saddle points \( SP_{N} \) approaches (as \( \theta \) varies) a pole of the spectral function \( u(\omega - \omega_d) \) with the location bounded away from the point where the two near first-order saddle points coalesce for \( \theta = \theta_1 \) to form a single second-order saddle point. The methods described in Appendixes B and C can then be applied to obtain the asymptotic approximations of \( I_z(z, \theta) \) and \([I^-(z, \theta) + I^+(z, \theta)]\) that are of the forms

\[
I_z(z, \theta) = A_B(z, t) + C_L(z, t) + R'(z, \theta), \quad 1 \leq \theta \leq \theta_1, \tag{2.22a}
\]

\[
I^-(z, \theta) + I^+(z, \theta) = A_B(z, t) + C_N^c(z, t) + R'(z, \theta), \quad \theta > \theta_1. \tag{2.22b}
\]

In both cases the expression for \( A_B(z, t) \) is the same as that in Eq. (2.20). Either quantity, \( C_L(z, t) \) or \( C_N^c(z, t) \), is asymptotically negligible if the corresponding saddle point does not approach a pole, in which case Eq. (2.22a) reduces to Eq. (2.20a) and/or Eq. (2.22b) reduces to Eq. (2.20b).

Combination of Eqs. (2.1)–(2.3) and (2.18)–(2.22) then leads to the following final general expression for the asymptotic approximation of the field \( A(z, t) \):

\[
A(z, t) = A_z(z, t) + A_B(z, t) + A_c(z, t) + R'(z, \theta). \tag{2.23}
\]

This representation is uniformly valid for all \( \theta \geq 1 \), provided that all the poles of the spectral function \( u(\omega - \omega_d) \) are bounded away from the limiting locations taken by \( SP_{D^p}(\theta) \) as \( \theta \to 1^+ \) and by \( SP_{N} \) as \( \theta \to \theta_1 \) from below. The contribution \( A_z(z, t) \) is obtained by adding all the terms that involve the poles of \( u(\omega - \omega_d) \), so that

\[
A_z(z, t) = -\text{Re}[2\pi i \lambda(\theta)] + C_D^-(z, t) + C_D^+(z, t) + C_L(z, t), \tag{2.24a}
\]

\[
1 \leq \theta \leq \theta_1,
\]

\[
A_z(z, t) = -\text{Re}[2\pi i \lambda(\theta)] + C_D^-(z, t) + C_D^+(z, t) + C_N^c(z, t), \tag{2.24b}
\]

\[
\theta > \theta_1.
\]

An estimate of the remainder term \( R'(z, \theta) \) as \( z \to \infty \) is obtained by taking the largest estimate of the remainder terms appearing in Eqs. (2.19)–(2.22).

An important feature of Eq. (2.23) is that the asymptotic behavior of the propagated plane-wave field \( A_z(z, t) \) is expressed as the sum of three terms that are essentially uncoupled so that they can be treated independently of one another. The dynamic behavior of \( A_z(z, t) \) is determined by the dynamic evolution of the distant saddle points \( SP_{D^p}^{\pm} \) and the value of the integrand at those saddle points. Since the distant saddle points are dominant first, \( A_z(z, t) \) describes the dynamic evolution of the first, or Sommerfeld, precursor field. It is asymptotically negligible during most of the second precursor and main signal evolution in the mature dispersion regime.

The dynamic behavior of \( A_B(z, t) \) is determined by the dynamic evolution of the near saddle points \( SP_{N} \) for \( 1 \leq \theta \leq \theta_1 \) and \( SP_{N}^{\pm} \) for \( \theta > \theta_1 \) as well as the value of the integrand at these saddle points. The near saddle point \( SP_{N} \) is of equal dominance with the distant saddle points \( SP_{D^p}^{\pm} \) at \( \theta = \theta_{SB} \) and remains dominant over the distant saddle points for \( \theta_{SB} < \theta \leq \theta_1 \), and for all \( \theta > \theta_1 \) the near saddle points \( SP_{N}^{\pm} \) remain dominant over the distant saddle points \( SP_{D^p}^{\pm} \), where

\[
\theta_{SB} \equiv \theta_0 - \frac{4\delta \omega^2}{3\theta_0 \omega^4} - \left[ \frac{27\delta \omega^2(\theta_0 - 1)^2}{4\theta_0 \omega^4} \right]^{1/3}
\]

\[
\times \left[ \left( 1 + \frac{2\delta \omega^2}{27\theta_0(\theta_0 - 1)\omega^4} \right)^{1/3} + 1 \right]^{1/3} - \left( 1 + \frac{2\delta \omega^2}{27\theta_0(\theta_0 - 1)\omega^4} \right)^{1/3} \right)^{1/3}. \tag{2.25}
\]

The contribution \( A_B(z, t) \) describes the dynamic evolution of the second, or Brillouin, precursor field. It is asymptotically negligible during most of the evolution of the first precursor and the main signal in the mature dispersion regime.

The dynamic behavior of \( A_c(z, t) \) is determined both by the poles of the spectral function \( u(\omega - \omega_d) \) and by the dynamic evolution of the saddle points that interact with these poles. The contribution \( A_c(z, t) \) is nonzero only if \( u(\omega - \omega_d) \) or \( \hat{f}(\omega) \) if Eq. (1.1) is used) has poles. If the envelope function \( u(t) \) of the field \( A(0, t) \) on the plane \( z = 0 \) is bounded for all time \( t \), then \( u(\omega - \omega_d) \) can have poles only if \( u(t) \) does not tend to zero too fast as \( t \to \infty \) [if \( u(t) \) is bounded and tends to zero with sufficient rapidity that the Fourier transform of \( u(t) \) converges uniformly for all real \( \omega \), then \( u(\omega - \omega_d) \) is an entire function of complex \( \omega \)]. Hence the implication of nonzero \( A_c(z, t) \) is that the field \( A(z, t) \) oscillates with angular frequency \( \omega_d \) for positive time \( t \) on the plane \( z = 0 \) and will tend to do the same at values \( z > 0 \) for large enough \( t \). As a result, the contribution \( A_c(z, t) \) describes the dynamic evolution of the main signal oscillating with fixed angular frequency \( \omega_d \). This contribution is asymptotically negligible during most of the precursor field evolution, provided that \( \omega_d < \omega_{SB} \), where

\[
\omega_{SB} \equiv \xi(\theta_{SB}) \approx \omega_0 \left( 2 + \frac{\delta \omega^2}{\omega_0^2} + \frac{5\delta \omega^2}{3\omega_0^2} \right)^{1/2}. \tag{2.26}
\]
When \( \omega > \omega_0 \) the distant saddle point \( \text{SP}_{+} \) passes in the vicinity of the pole when it is the dominant saddle point and the associated pole contribution then first becomes dominant over the Sommerfeld precursor field at a value \( \theta_1 \) between 1 and \( \theta_0 \). The Brillouin precursor field then becomes dominant over this pole contribution at a value \( \theta_2 \) between \( \theta_B \) and \( \theta_0 \). This pole contribution then becomes dominant over the second precursor field at a value \( \theta > \theta_0 \) and remains dominant for all larger values of \( \theta \). In this case the contribution \( \varphi(z, t) \) describes the dynamic evolution of the signal oscillating at \( \omega \), that is asymptotically dominant over a short space–time interval between the individual Sommerfeld and Brillouin precursor field evolutions and is again asymptotically dominant after the Brillouin precursor field evolution.\(^{4,5} \) The signal evolution over the interval \( \theta_1 < \theta < \theta_2 \) is called the prepulse, and the remaining signal evolution for \( \theta > \theta_2 \) is called the main signal.

For most values of \( \theta \), only one of the terms \( \varphi(z, t) \) and \( \varphi(z, t) \) appearing in Eq. (2.23) is important at a time. There are short intervals of \( \theta \), however, during which two or more of these terms are significant for fixed values of the propagation distance \( z \). These intervals mark the transition periods in which the field is changing its character from one form to another, and the presence of both terms in the expression leads to a continuous transition in the behavior of the field, as described in Appendix D below. As a consequence, Eq. (2.23) displays the entire evolution of the propagated field through its various components in a continuous manner.

### 3. Uniform Asymptotic Description of the Sommerfeld Precursor Field

The contributions of the two distant saddle points to the asymptotic behavior of the field [Eq. (1.7)] for sufficiently large values of the propagation distance \( z \) yield the dynamic evolution of the first, or Sommerfeld, precursor field. From the theorem of Appendix A and the results of Section 2, the integral in Eq. (1.7), when taken over the two contours \( P_{+}(\theta) \) and \( P_{-}(\theta) \), yields the uniform asymptotic approximation

\[
 \varphi(z, t) \sim \left( \frac{1}{2} \right)^{1+2} \frac{\xi^{1/2}(\theta)}{b} \times \left( \frac{\xi(\theta)}{\theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]}} \right)^{1/2} \times (\theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]})^{1/2},
\]

(3.1)

for all \( \theta \geq 1 \). The remainder term \( R(z, \theta) \) is bounded by the inequality (A7) for all \( \theta \geq 1 \) with the constant \( K \) independent of \( \theta \). The real parameter \( \nu \) is defined by Eq. (A4). Furthermore, from the defining relations, Eqs. (A8)–(A11) and (2.8a) and (2.8b), the coefficients appearing in this uniform expansion are given by

\[
 \alpha(\theta) \equiv \xi(\theta) \left( \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]} \right),
\]

(3.2)

\[
 \beta(\theta) \equiv -ib \left( [1 + \eta(\theta)](\theta - 1) + \frac{(1/2)b^2[1 - \eta(\theta)]}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]} \right),
\]

(3.3)

All the square-root expressions appearing in Eqs. (3.4) and (3.5) are real and positive and have been chosen so as to satisfy the limiting Eq. (A15).

Substitution of relations (3.2)–(3.5) into Eq. (3.1) then yields

\[
 \varphi(z, t) \sim \left( \frac{1}{2} \right)^{1+2} \frac{\xi^{1/2}(\theta)}{b} \times \left( \frac{\xi(\theta)}{\theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]}} \right)^{1/2} \times \text{Re} \exp \left( -\frac{i}{2} \sum_{\theta > 1} \left( \frac{1}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]} \right) \right)
\]

(3.6)

as \( z \to \infty \) uniformly for all \( \theta \geq 1 \). This result constitutes the general expression for the uniform asymptotic behavior of the Sommerfeld precursor field. It reduces to the nonuniform expansion given in Eq. (4.21) of Ref. 5 for \( \theta \geq 1 + \Delta \), where \( \Delta > 0 \).

**A. The Delta-Function Field**

For the delta-function pulse the initial spectrum is given by \( \tilde{\varphi}(\omega - \omega_0) = -i \), so that \( \nu = -1 \). As is pointed out in Appendix A, the uniform asymptotic expansion remains valid for all \( \theta \geq 1 \) when \( \nu < 0 \), provided that its limiting behavior is finite as \( \theta \) approaches unity from above. With these substitutions the asymptotic expression in relation (3.6) becomes
1.5
1.0
0.5

A,(zt)
(units of x^2)

Fig. 2. Uniform asymptotic description of the dynamic evolution of the Sommerfeld precursor field A,(z, t) for an input delta-function pulse at a propagation distance of z = 1 x 10^{-4} cm in a highly absorptive and dispersive medium.

A,(z, t) \sim \frac{\xi(\theta)}{2b} \left( \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right)^{1/2}
\times \exp \left( -\frac{-z}{c} \left[ (1 + \eta(\theta))(\theta + 1) + \frac{(1/2)b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right] \right)
\times \left[ 2\xi(\theta)J_{-1/2}(\xi(\theta)) \left( \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right) \right]
+ 3\delta[1 - \eta(\theta)]J_{-1/2}(\xi(\theta)) \left( \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right).

(3.7)

Since this expression vanishes in the limit as \theta \to 1^+, it is a uniformly valid approximation of the first precursor field for the delta-function pulse for all \theta \geq 1. The dynamical evolution of this field structure, as given by relation (3.7), is illustrated in Fig. 2 at a fixed propagation distance of z = 1 x 10^{-4} cm. The medium parameters used here and throughout this paper (with the exception of one specific example, which is noted explicitly) have the values

\omega_0 = 4.0 \times 10^{16}/\text{sec},

b^2 = 20.0 \times 10^{22}/\text{sec}^2,

\delta = 0.28 \times 10^{16}/\text{sec},

which correspond to a highly absorptive and dispersive medium.

B. The Unit-Step-Function-Modulated Signal
For an input unit-step-function-modulated signal with a fixed carrier frequency \omega_c, the spectrum of the initial field envelope is given by Eq. (1.10b), so that \nu = 0 and

\tilde{G}(\omega_{\text{SP}} - \omega_c) \approx \frac{i}{\pm \xi(\theta) - \omega_c - \delta[1 + \eta(\theta)].

(3.8)

The uniform asymptotic expansion of the Sommerfeld precursor field in this case becomes

A,(z, t) \sim \frac{\xi(\theta)}{2b} \left( \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right)^{1/2}
\times \exp \left( -\frac{-z}{c} \left[ (1 + \eta(\theta))(\theta - 1) + \frac{(1/2)b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right] \right)
\times \left[ \frac{\delta}{2} \left[ \frac{\xi(\theta)[5 - \eta(\theta)] + 3\omega_c[1 - \eta(\theta)]}{\xi^2(\theta) + \delta^2[1 + \eta(\theta)]^2} \right] \right]
- \frac{\xi(\theta)[5 - \eta(\theta)] - 3\omega_c[1 - \eta(\theta)]}{\xi^2(\theta) + \omega_c^2 + \delta^2[1 + \eta(\theta)]^2}
\times \exp \left( \frac{-z}{c} \left[ \frac{\omega_c[5 - \eta(\theta)]}{\xi^2(\theta) + \delta^2[1 + \eta(\theta)]^2} \right] \right)
+ \frac{\xi(\theta)[\xi(\theta) - \omega_c] - (3/2)\delta^2[1 - \eta^2(\theta)]}{\xi^2(\theta) + \omega_c^2 + \delta^2[1 + \eta(\theta)]^2}
- \frac{\xi(\theta)[\xi(\theta) + \omega_c] - (3/2)\delta^2[1 - \eta^2(\theta)]}{\xi^2(\theta) + \omega_c^2 + \delta^2[1 + \eta(\theta)]^2}
\times \exp \left( \frac{-z}{c} \left[ \frac{\omega_c[5 - \eta(\theta)]}{\xi^2(\theta) + \delta^2[1 + \eta(\theta)]^2} \right] \right)
\times \exp \left( \frac{-z}{c} \left[ \frac{\omega_c[5 - \eta(\theta)]}{\xi^2(\theta) + \delta^2[1 + \eta(\theta)]^2} \right] \right).

(3.9)

as z \to \infty uniformly for all \theta \geq 1. It is evident from this result that A,(z, t), and consequently A,(z, t), asymptotically vanishes at \theta = 1 but is nonzero for \theta = 1 + \epsilon where \epsilon > 0 can be arbitrarily small. Consequently, the front of the Sommerfeld precursor field travels with the velocity of light c in vacuum for an input unit-step-function-modulated signal as well as for an input delta-function pulse. The dynamic evolution with \theta of this field structure, as given in relation (3.9), is illustrated in Fig. 3 at a fixed propagation distance of z = 1 x 10^{-4} cm with applied signal frequency \omega_c = 1 \times 10^{16}/\text{sec}, which is well below the medium resonance frequency \omega_0. Comparison of Figs. 2 and 3 shows that the initial rise in amplitude of the Sommerfeld precursor is more rapid for the case of the delta-function pulse than it is for the unit-step-function-modulated signal (notice the change in scale between these two figures). The instantaneous angular fre-
quency of oscillation of $A_0(z, t)$ is the same as that given in the nonuniform theory.\footnote{5}

4. UNIFORM ASYMPTOTIC DESCRIPTION OF THE BRILLOUIN PRECURSOR FIELD

The contributions of the near saddle points to the asymptotic behavior of the field $A(z, t)$ for sufficiently large values of the observation distance $z$ yield the dynamic evolution of the second, or Brillouin, precursor field. This contribution to the total asymptotic behavior of $A(z, t)$ is denoted by $A_B(z, t)$ and is dominant over the Sommerfeld precursor field for all $\theta > \theta_{SB}$.

The required uniform asymptotic approximation is obtained by direct application of the theorem given in Appendix B. To begin, the appropriate path $L_i$ that the path $P(\theta)$ is mapped into under the cubic transformation [Eq. (B6)] must first be determined. From Fig. 7 of Ref. 5 it is seen that the angle of slope of the path $P(\theta)$ as it leaves the second-order saddle point at $\omega = \omega_{SB}$ when $\theta = \theta_{SB}$ is given by $\alpha = \pi/6$. If the quantity $\Delta \omega$ appearing in Eq. (B24) is taken to lie along the portion of the path $P(\theta)$ that approaches $\omega_{SB}$ from the left half of the complex $\omega$ plane, then $\arg(\Delta \omega) = 5\pi/6$. Hence Eq. (B24) states that the argument of $\Delta \omega$ that lies along the corresponding portion of the transformed contour is $\arg(\Delta \omega) = 2\pi/3$, and this shows that the transformed contour originates in region 2 of Fig. 11 below (see Appendix B). If $\Delta \omega$ is taken to lie along the portion of $P(\theta)$ that departs from $\omega_{SB}$ toward the right half of the complex $\omega$ plane, then $\arg(\Delta \omega) = \pi/6$. Hence Eq. (B24) states that the argument of $\Delta \nu$ that lies along the corresponding portion of the transformed contour is $\arg(\Delta \nu) = 0$ so that the transformed contour terminates in region 1 of Fig. 11. Consequently, the transformation [Eq. (B6)] combined with Eq. (B21) transforms the contour $P(\theta)$ into an $L_i$ path so that the function $\phi(\theta)$ appearing in Eq. (B4) is given by Eq. (B23b).

Although the theorem given in Appendix B is directly applicable to the present problem for all $\theta > 1$, it is necessary to treat the two cases $1 < \theta < \theta_1$ and $\theta \geq \theta_1$ separately because the approximate expressions [relations (2.9a) and (2.9c)] for the near-saddle-point locations differ in these two cases. Nonetheless, the results for the two cases combined are applicable to the present problem for all $\theta > 1$.

Consider first the uniform asymptotic behavior of the second precursor field for values of $\theta$ in the domain $1 < \theta \leq \theta_1$. In this case the two near-saddle-point locations are given by relations (2.9a) and (2.9b), so that the approximate phase behavior at these locations is given by

\[
\phi(\omega_{SP}, \theta) = \frac{2}{3} \delta(\theta) - |\psi(\theta)| \left( \theta_0 - \theta \right) - \frac{b^2}{2\theta_0 \omega_0^4} \left[ \frac{2}{3} \delta(\theta) + |\psi(\theta)| \right] \left[ \frac{2}{3} \alpha \delta(\theta) - a|\psi(\theta)| - 2\beta \right].
\] (4.1)

\[
\phi(\omega_{SP}, \theta) = \frac{2}{3} \delta(\theta) + |\psi(\theta)| \left( \theta_0 - \theta \right) - \frac{b^2}{2\theta_0 \omega_0^4} \left[ \frac{2}{3} \delta(\theta) + |\psi(\theta)| \right] \left[ \frac{2}{3} \alpha \delta(\theta) + a|\psi(\theta)| - 2\beta \right].
\] (4.2)

where $\omega_{SP_1} = \omega_{SP_{\alpha+}}$ and $\omega_{SP_2} = \omega_{SP_{\alpha-}}$. Application of the theorem in Appendix B then yields the asymptotic approximation

\[
A_B(z, t) = -\text{Re} \left[ \exp \left( \frac{z}{c} \omega_{0}(\theta) \right) \left( \frac{c}{z} \right)^{1/3} \exp \left( -\frac{2\pi i}{3} \right) \right] \times A_i \left[ \frac{1}{2} \left( \frac{\omega_{SP} - \omega_{c}}{\omega_{SP} - \omega_{c}} \right) h_1(\theta) + \frac{\omega_{SP} - \omega_{c}}{\omega_{SP} - \omega_{c}} h_2(\theta) \right] + \phi \left( \frac{1}{\alpha} \right)
\]

\[
\times \left\{ \left[ \frac{1}{2} \left( \frac{\omega_{SP} - \omega_{c}}{\omega_{SP} - \omega_{c}} \right) h_1(\theta) - \frac{\omega_{SP} - \omega_{c}}{\omega_{SP} - \omega_{c}} h_2(\theta) \right] \right\} \right\}^{1/3}
\]

as $z \to \infty$ uniformly for $1 < \theta \leq \theta_1$. From Eqs. (B8)-(B10) and relations (4.1) and (4.2), the coefficients appearing in this expression are found to be given by

\[
\alpha_{0}(\theta) = -\frac{2}{3} \delta(\theta) - \theta_0 - \frac{b^2}{3 \omega_0^4} \left( \frac{3}{2} \theta_0 - \theta_0 \right) + \frac{\omega_{SP} - \omega_{c}}{\omega_{SP} - \omega_{c}} \left( \frac{3}{2} \theta_0 - \theta_0 \right) - \frac{b^2}{3 \omega_0^4} \left( \frac{3}{2} \theta_0 - \theta_0 \right)
\]

\[
\alpha_{1}(\theta) = \frac{1}{2} \left( \frac{\omega_{SP} - \omega_{c}}{\omega_{SP} - \omega_{c}} \right) h_1(\theta) + \frac{\omega_{SP} - \omega_{c}}{\omega_{SP} - \omega_{c}} h_2(\theta) \right] \right\}^{1/3},
\] (4.5)

\[
h_{1,2}(\theta) = \left\{ \frac{2\theta_0 \omega_0^4}{b^3 \alpha \omega(\theta) \pm 2 \left[ 1 - \alpha(\theta) \right]} \left( \frac{2\theta_0 \omega_0^4}{b^3 \alpha \omega(\theta) \pm 2 \left[ 1 - \alpha(\theta) \right]} \right)^{3/4} \right\}^{1/3}.
\] (4.6)

for all values of $\theta$ in the domain $1 < \theta \leq \theta_1$. The upper sign in relation (4.6) corresponds to $h_1(\theta)$, and the lower sign corresponds to $h_2(\theta)$. Furthermore, from Eq. (B11), the preceding expression reduces in the limit as $\theta$ approaches $\theta_1$ from below to

\[
h_1(\theta) = \lim_{\theta \to \theta_1^-} (h_{1,2}(\theta)) \approx -\frac{2\theta_0 \omega_0^4}{3 \alpha b^2} \left( \frac{2\theta_0 \omega_0^4}{3 \alpha b^2} \right)^{1/3}.
\] (4.7)

The proper values of the multivalued functions appearing in relations (4.5)-(4.7) are determined by the conditions presented in Appendix B. In particular, the phase of $h_{1,2}(\theta)$ is specified by Eq. (B14), so that

\[
\lim_{\theta \to \theta_1^-} \left| \arg[h_{1,2}(\theta)] \right| = \alpha^+,
\] (4.8)

where $\alpha^+$ is the angle of slope of the path of steepest descent as it leaves the second-order saddle point at $\omega_{SP} = \omega_{SP}(\theta)$ for $\theta = \theta_1$. From Ref. 5 it is seen that $\alpha^+ = \pi/6$, so that the argument of $h_1(\theta)$ is $\pi/6$. Moreover, since the quantity within the bold parentheses in relation (4.6) is real and
negative for all $\theta$ in the domain $1 < \theta < \theta_1$, then the argument of $h_{12}(\theta)$ is independent of $\theta$ over that domain. Hence

$$\arg[h_{12}(\theta)] = \frac{\pi}{6}$$  \hspace{1cm} (4.9)$$
for $1 < \theta < \theta_1$.

The proper value of the phase of $\alpha_1^{1/3}(\theta)$ is determined by Eq. (B.16) with $n = 0$, which may be rewritten as

$$\lim_{\theta \to \theta_1^-} [\arg[\alpha_1^{1/3}(\theta)]] = \tilde{\alpha}_{12} - \tilde{\alpha}^+,$$  \hspace{1cm} (4.10)$$
where $\tilde{\alpha}_{12}$ is the angle of slope of the vector from the saddle point $SP_2$ to the saddle point $SP_1$ in the complex $\sigma$ plane and $\tilde{\alpha}^+ = \pi/6$. Since $\tilde{\alpha}_{12} = \pi/2$ for $1 < \theta < \theta_1$, then

$$\lim_{\theta \to \theta_1^-} [\arg[\alpha_1^{1/3}(\theta)]] = \pi/3.$$  \hspace{1cm} (4.11)$$
Moreover, since the quantity within the bold parentheses in relation (4.5) is real and negative for all $\theta$ in the domain $1 < \theta < \theta_1$, then the argument of $\alpha_1^{1/3}(\theta)$ is independent of $\theta$ in that domain. Hence

$$\arg[\alpha_1^{1/3}(\theta)] = \pi/3$$  \hspace{1cm} (4.12)$$
for $1 < \theta < \theta_1$.

Since $\arg[\alpha_1(\theta)] = 2\pi/3$, the argument of the Airy function and its first derivative appearing in Eq. (4.3) is real and nonnegative for $1 < \theta < \theta_1$. Application of the values of the arguments of $h_{12}(\theta)$ and $\alpha_1(\theta)$ obtained above simplifies the asymptotic expression given in Eq. (4.3) to read as

$$A_B(z, t) \sim \exp\left\{ \frac{z}{c} \alpha_0(\theta) \right\} \left\{ \frac{1}{2} \left( \frac{c^3}{\sigma_{SP_1}} \right)^{1/3} \right\} \Re[\bar{u}(\omega_{SP_1} - \omega) h_1(t)]$$

$$+ \bar{u}(\omega_{SP_1} - \omega) h_2(\theta)]A_1 \left[ \alpha_1(\theta) \left( \frac{z}{c} \right)^{2/3} \right]$$

$$- \bar{u}(\omega_{SP_2} - \omega) h_2(\theta)]A_1 \left[ \alpha_1(\theta) \left( \frac{z}{c} \right)^{2/3} \right]$$  \hspace{1cm} (4.13)$$
as $z \to \infty$ uniformly for all $\theta$ in the domain $1 < \theta < \theta_1$. The limiting form of this expression in the limit as $\theta$ approaches the critical value $\theta_1$ from below is given by

$$A_B(z, t) \sim \frac{\alpha_0}{2\pi \sqrt{3}} \Gamma\left( \frac{1}{3} \right) \left( \frac{2\theta_1 \omega_0 c^3}{3b^2 z^3} \right) \Re[\bar{u}(\omega_{SP_1} - \omega_0)]$$

$$\times \exp\left\{ \frac{2\theta_1 \omega_0 c^3}{3bc^2 z^3} \right\} \left( \theta - \theta_1 + \frac{4z^2 b^2}{9ab^2 \omega_0^4} \right)$$  \hspace{1cm} (4.14)$$
as $z \to \infty$ for $\theta = \theta_1 = ct_1/z$.

For a numerical evaluation of the second precursor field $A_B(z, t)$ for values of $\theta$ in the domain $(1, \theta_1]$, the asymptotic expression given in relation (4.13) is useless for values $1 < \theta > 0$ that are too small because of the numerical instabilities that result from the indeterminate form of the equation that takes as $\theta$ approaches $\theta_1$ from below. As $\theta$ approaches $\theta_1$ from below, the limiting forms given in Eqs. (B11)-(B13) may be used in the uniform expansion [Eq. (4.3)]. Moreover, for values of $\theta$ close to $\theta_1$, the second term in Eq. (4.3) is negligible in comparison to the first and may then be neglected. With these substitutions one obtains the following asymptotic expression for the second precursor field:

$$A_B(z, t) \sim \exp\left\{ \frac{z}{c} \alpha_0(\theta) \right\} \left\{ \frac{1}{2} \left( \frac{c^3}{\sigma_{SP_1}} \right)^{1/3} \right\} \Re[\bar{u}(\omega_{SP_1} - \omega) h_1(t)]$$

$$\times \Re[\bar{u}(\omega_{SP_1} - \omega_0) + \bar{u}(\omega_{SP_1} - \omega_0)]A_1 \left[ \alpha_1(\theta) \left( \frac{z}{c} \right)^{2/3} \right]$$  \hspace{1cm} (4.15)$$
as $z \to \infty$ and $\theta \to \theta_1^-$. At $\theta = \theta_1$, this expression reduces to that given in Eq. (4.14). For numerical calculations, the asymptotic expressions given in relations (4.13) and (4.15) yield a uniform, continuous evolution of the second precursor field for all values of $\theta$ in the domain $(1, \theta_1]$ when relation (4.13) is restricted to the domain $(1, \theta_1]$ and relation (4.15) is restricted to the domain $(\theta_0, \theta_1]$, as shown in Ref. 4. Since the argument of the Airy function and its first derivative are real and positive for $\theta \in (1, \theta_1]$, the second precursor field is nonoscillatory for these values of $\theta$.

Consider now the uniform asymptotic behavior of the Brillouin precursor for values of $\theta$ in the domain $\theta > \theta_1$. In this case, the two near-saddle-point locations are given by relations (2.9b) and (2.9c), and the approximate phase behavior at these points is given by relations (2.16) and (2.17). Application of the theorem in Appendix B then yields the asymptotic approximation

$$A_B(z, t) \sim -\Re[\exp\left\{ \frac{z}{c} \alpha_0(\theta) \right\} \left\{ \frac{1}{2} \left( \frac{c^3}{\sigma_{SP_1}} \right)^{1/3} \right\} \Re[\bar{u}(\omega_{SP_1} - \omega) h_1(t)]$$

$$+ \bar{u}(\omega_{SP_1} - \omega) h_2(\theta)]A_1 \left[ \alpha_1(\theta) \left( \frac{z}{c} \right)^{2/3} \right]$$

$$+ \bar{u}(\omega_{SP_1} - \omega) h_2(\theta)]A_1 \left[ \alpha_1(\theta) \left( \frac{z}{c} \right)^{2/3} \right]$$  \hspace{1cm} (4.16)$$
as $z \to \infty$ uniformly for $\theta \geq \theta_1$. From Eqs. (B8)-(B10) and relations (2.17a) and (2.17b), the coefficients appearing in this expression are found to be given by

$$\alpha_0(\theta) \approx -\delta \left\{ \frac{2}{3} \zeta(\theta) \right\} \theta - \theta_0 + b^2 \right\} \left\{ \frac{1}{2} \right\} \left( \theta - \theta_0 + b^2 \right\}$$

$$\alpha_1(\theta) \approx \left\{ \frac{3}{2} \right\} \left( \theta - \theta_0 + \frac{b^2}{2\theta_0 \omega_0^4} \right)^{1/3}$$

$$\times \left\{ \frac{3}{2} \right\} \left( \theta - \theta_0 + \frac{b^2}{2\theta_0 \omega_0^4} \right)^{1/6}$$  \hspace{1cm} (4.17)$$
for all values of $\theta$ in the domain $\theta \geq \theta_1$. In the limit as $\theta$ approaches $\theta_1$ from above, relation (4.19) is replaced by its limiting form [Eq. (B11)], so that

$$\lim_{\theta \to \theta_1^+} \alpha_0(\theta) = \frac{2}{3} \zeta(\theta) \theta - \theta_0 + b^2 \right\} \left\{ \frac{1}{2} \right\} \left( \theta - \theta_0 + b^2 \right\}$$

$$\times \left\{ \frac{3}{2} \right\} \left( \theta - \theta_0 + \frac{b^2}{2\theta_0 \omega_0^4} \right)^{1/3}$$

$$\times \left\{ \frac{3}{2} \right\} \left( \theta - \theta_0 + \frac{b^2}{2\theta_0 \omega_0^4} \right)^{1/6}$$  \hspace{1cm} (4.18)$$
for all values of $\theta$ in the domain $\theta \geq \theta_1$. In the limit as $\theta$ approaches $\theta_1$ from above, relation (4.19) is replaced by its limiting form [Eq. (B11)], so that

$$\lim_{\theta \to \theta_1^+} \alpha_1(\theta) = \frac{3}{2} \zeta(\theta) \theta - \theta_0 + b^2 \right\} \left\{ \frac{1}{2} \right\} \left( \theta - \theta_0 + b^2 \right\}$$

$$\times \left\{ \frac{3}{2} \right\} \left( \theta - \theta_0 + \frac{b^2}{2\theta_0 \omega_0^4} \right)^{1/3}$$

$$\times \left\{ \frac{3}{2} \right\} \left( \theta - \theta_0 + \frac{b^2}{2\theta_0 \omega_0^4} \right)^{1/6}$$  \hspace{1cm} (4.19)$$
The proper values of the multivalued functions appearing in relations (4.17)-(4.20) are determined by the conditions presented in Appendix B. In particular, the phase of \( h^+(\theta) \) is specified by Eq. (B14), so that

\[
\lim_{\theta \to \theta_1^+} [\arg [h^+(\theta)]] = \tilde{\alpha}_1^+,
\]

where \( \tilde{\alpha}_1^+ = \pi/6 \) is the angle of slope of the path of steepest descent as it leaves the second-order saddle point at \( \omega_{SP^N} \) for \( \theta = \theta_1 \). The above relation shows that the argument of \( h(\theta) \) is \( \pi/6 \). Moreover, since the quantity within the bold square brackets in relation (4.19) is real and negative for all \( \theta \geq \theta_1 \), the argument of \( h^+(\theta) \) is independent of \( \theta \) over that domain. Hence

\[
\arg[h^+(\theta)] = \pi/6 \quad (4.22)
\]

for \( \theta \geq \theta_1 \).

The proper value of the phase of \( \alpha_1^{1/2}(\theta) \) is determined by Eq. (B16) with \( n = 0 \), which may be rewritten as

\[
\lim_{\delta \to \delta_1^+} [\arg [\alpha_1^{1/2}(\theta)]] = \tilde{\alpha}_1 \tilde{\alpha}_1^+ = \tilde{\alpha}_1^+ = \pi/6,
\]

where \( \tilde{\alpha}_1 \) is the angle of slope of the vector from the saddle point \( SP^-\) to the saddle point \( SP^+ \) in the complex plane and \( \tilde{\alpha}_1^+ = \pi/6 \). Since \( \tilde{\alpha}_1 \) is \( 0 \) for \( \theta \geq \theta_1 \), then

\[
\lim_{\delta \to \delta_1^+} [\arg [\alpha_1^{1/2}(\theta)]] = -\pi/6.
\]

Moreover, since the quantity within the bold square brackets in relation (4.18) is negative imaginary for all \( \theta \geq \theta_1 \), the argument of \( \alpha_1^{1/2}(\theta) \) is determined above then simplifies the asymptotic expression in Eq. (4.16) to

\[
A_B(z, t) \sim \exp \left[ \frac{z}{c} \alpha_0(\theta) \right] \frac{c^{1/3}}{2} \left( \frac{z}{c} \right)^{1/3} \left[ \frac{1}{2} \left( \frac{z}{c} \right)^{1/3} \right] \frac{c}{\alpha_1(\theta)} \left( \frac{z}{c} \right)^{2/3} \left[ \frac{1}{2} \left( \frac{z}{c} \right)^{1/3} \right]
\]

\[
+ \left[ \frac{2}{2 \alpha_1(\theta)} \right] \frac{c}{\alpha_1(\theta)} \left( \frac{z}{c} \right)^{2/3} \left[ \frac{1}{2} \left( \frac{z}{c} \right)^{1/3} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
+ \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
\times \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
\times A_1 \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
\text{as } z \to \infty \text{ uniformly for all } \theta \geq \theta_1. \quad \text{The limiting form of this expression in the limit as } \theta \text{ approaches the critical value } \theta_1 \text{ from above is given simply by relation (4.14).}
\]

For a numerical evaluation of the second precursor field \( A_B(z, t) \) for values of \( \theta \geq \theta_1 \), the asymptotic expression given in relation (4.26) is useless for values \( \theta - \theta_1 > 0 \) that are too small because of the numerical instabilities that result from the indeterminate form that the equation takes as \( \theta \) approaches \( \theta_1 \) from above. As \( \theta \) approaches the critical value \( \theta_1 \) from above, the limiting forms given in Eqs. (B11)–(B13) may be used in the uniform expansion [Eq. (4.16)]. Moreover, for values of \( \theta \) close to \( \theta_1 \), the second term appearing in Eq. (4.16) is negligible in comparison with the first and may be neglected. With these substitutions, the following asymptotic expression for the second precursor field is obtained:

\[
A_B(z, t) \sim \exp \left[ \frac{z}{c} \alpha_0(\theta) \right] \frac{c^{1/3}}{2} \left( \frac{z}{c} \right)^{1/3} \left[ \frac{1}{2} \left( \frac{z}{c} \right)^{1/3} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
+ \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
\times \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
\times \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right] \left[ \frac{2}{2 \alpha_1(\theta)} \right]
\]

\[
\text{as } z \to \infty \text{ uniformly for all } \theta \text{ in the domain } 1 < \theta \leq \theta_1. \quad \text{The functions } \alpha_0(\theta) \text{ and } \alpha_1(\theta) \text{ appearing in this expression are given by relations (4.4) and (4.5), respectively. For a numerical evaluation of } A_B(z, t), \text{ relation (4.29) is unstable and}
\]
not useful for \( \theta \in [\theta_0, \theta_1] \). For that purpose relation (4.15) is applicable and yields

\[
A_B(z, t) \sim \left( \frac{2\theta_0 \omega_0^4 c}{3 \alpha_0 b^2 z} \right)^{1/3} A_1 \left[ -a_1(\theta) \left( \frac{z}{c} \right)^{2/3} \right] \times \exp \left[ \frac{2a_2}{3\alpha_0} \left( \theta_0 - \frac{4\delta^2 b_2^4}{9\alpha_0 \omega_0^4} - \theta \right) \right]
\]

as \( z \to \infty \) for \( \theta_0 \leq \theta \leq \theta_1 \). At \( \theta = \theta_1 \) both of these expressions reduce to

\[
A_B(z, t) \sim \left( \frac{\Gamma(z)}{2\pi \sqrt{3}} \right)^{1/3} \left( \frac{2\theta_0 \omega_0^4 c}{3 \alpha_0 b^2 z} \right)^{1/3} \times \exp \left[ \frac{2a_2}{3\alpha_0} \left( \theta - \frac{4\delta^2 b_2^4}{9\alpha_0 \omega_0^4} - \theta_1 \right) \right]
\]

as \( z \to \infty \) for \( \theta = \theta_1 = \text{ct}/z \).

For values of \( \theta \geq \theta_1 \), the uniform asymptotic behavior of the second precursor field is given by relation (4.26), which, for numerical calculations, reduces to relation (4.27) for \( \theta \in [\theta_1, \theta_2] \). With \( \eta = \omega - \omega_0 \) for these two equations become, in reverse order,

\[
A_B(z, t) \sim \left( \frac{2\theta_0 \omega_0^4 c}{3 \alpha_0 b^2 z} \right)^{1/3} A_2 \left[ -a_1(\theta) \left( \frac{z}{c} \right)^{2/3} \right] \times \exp \left[ -\frac{2a_2}{3\alpha_0} \left( \theta - \frac{4\delta^2 b_2^4}{9\alpha_0 \omega_0^4} \right) \right]
\]

as \( z \to \infty \) for \( \theta_1 \leq \theta \leq \theta_2 \) and

\[
A_B(z, t) \sim \frac{\omega_0^2}{b} \left( \frac{2\theta_0}{3 \alpha_0 \psi(\theta)} \right)^{1/3} A_1 \left[ -a_1(\theta) \left( \frac{z}{c} \right)^{1/4} \right] \times \exp \left[ \frac{\omega_0^2}{3 \alpha_0 \psi(\theta)} \right] A_2 \left[ -a_1(\theta) \left( \frac{z}{c} \right)^{1/4} \right]
\]

as \( z \to \infty \) uniformly for all \( \theta \geq \theta_1 \). The functions \( a_0(\theta) \) and \( a_1(\theta) \) appearing in these two expressions are given by relations (4.17) and (4.18), respectively. For a numerical evaluation of \( A_B(z, t) \), relation (4.33) is unstable and not useful for \( \theta \in [\theta_1, \theta_2] \) and must be replaced by the expression given in relation (4.32). Both of these expressions reduce to relation (4.31) at \( \theta = \theta_1 \).

The dynamic evolution of the Brillouin precursor field structure for the delta-function pulse, as given by the asymptotic expressions in relations (4.29)–(4.33), is illustrated in Fig. 4 for a fixed propagation distance of \( z = 1 \times 10^{-4} \) cm. Notice that the field amplitude of the Brillouin precursor is nearly zero for \( 1 < \theta < \theta_{\text{SP}} \), reaches a peak amplitude near \( \theta = \theta_0 \) at this space–time point there is no exponential attenuation of the field, and then decays exponentially with increasing \( \theta > \theta_0 \). For \( \theta > \theta_1 \) the field oscillates with an oscillation frequency that increases monotonically with increasing \( \theta \) and approaches \( (\omega_0^2 - \delta^2)^{1/2} \) as \( \theta \) increases to infinity.

B. The Unit-Step-Function-Modulated Signal

For the unit-step-function-modulated signal with fixed carrier frequency \( \omega_\text{c} \), the spectrum of the initial field envelope is given by Eq. (1.10b), so that

\[
\tilde{u}(\omega_\text{SP}, \omega) \approx \frac{[\psi(\theta) - (2/3)\delta(\theta) - j\omega_c]}{\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]}, \quad (4.34a)
\]

\[
\tilde{u}(\omega_\text{SP}, \omega) \approx \frac{[\psi(\theta) + (2/3)\delta(\theta) - j\omega_c]}{\omega_c^2 + [(\psi(\theta) + (2/3)\delta(\theta))^2]}, \quad (4.34b)
\]

for \( 1 < \theta \leq \theta_1 \) and

\[
\tilde{u}(\omega, \omega) \approx \frac{[\pm(\psi(\theta) - \omega_c)]^2 + [(4/9)\delta^2(\delta(\theta))^2]}{[\psi(\theta) - \omega_c]^2} \quad (4.35)
\]

for \( \theta \geq \theta_1 \).

For values of \( \theta \) in the domain \( 1 < \theta \leq \theta_1 \), the uniform asymptotic behavior of the second precursor field is given by relation (4.13). Substitution of relation (4.34a) into that expression then yields

\[
A_B(z, t) \sim \frac{\omega_0^2}{2b} \left( \frac{z}{2} \right)^{1/3} \exp \left[ \frac{\omega_0^2}{2b} \right]
\]

\[
\times \left[ \left( \frac{\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]}{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]} \right)^{1/2} \right] \times \left[ \left( \frac{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]}{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]} \right)^{1/2} \right]
\]

\[
\times \frac{2\theta_0}{3\alpha_0 \psi(\theta)} \left[ \frac{a_1(\theta)(\frac{z}{c})^{2/3}}{[\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]} \right]^2 \times \left( \frac{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]}{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]} \right)^{1/2} \times \left( \frac{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]}{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]} \right)^{1/2} \times \left( \frac{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]}{3(\omega_c^2 + [(\psi(\theta) - (2/3)\delta(\theta))^2]} \right)^{1/2} \times A_1 \left[ a_1(\theta)(\frac{z}{c})^{2/3} \right] \times A_2 \left[ a_1(\theta)(\frac{z}{c})^{1/4} \right]
\]

as \( z \to \infty \) uniformly for all \( \theta \) in the domain \( 1 < \theta \leq \theta_1 \). The functions \( a_0(\theta) \) and \( a_1(\theta) \) appearing in this expression are given by relations (4.4) and (4.5), respectively. For a numerical evaluation of \( A_B(z, t) \), relation (4.36) is unstable and not useful for \( \theta \in [\theta_0, \theta_1] \). For that purpose relation (4.15) is applicable and yields
The dynamic evolution of the second precursor field structure for the unit-step-function-modulated signal with carrier frequency \( \omega_c = 1 \times 10^{16} \text{sec}^{-1} \) at a propagation distance of \( z = 1 \times 10^{-3} \text{cm} \) in a highly absorptive and dispersive medium.

5. UNIFORM ASYMPTOTIC DESCRIPTION OF THE SPECTRAL POLE CONTRIBUTION

The contribution \( A_p(z, t) \) to the asymptotic behavior of the propagated field that is due to the presence of any simple pole singularities of the spectral function \( \hat{u}(\omega - \omega_c) \) is now considered. The field component \( A_p(z, t) \) is associated with any long-term signal in the propagated field structure. The case of special interest in the present paper is that for which there is a single pole singularity \( \omega_p \) of the spectral amplitude function \( \hat{u}(\omega - \omega_c) \) that is real and positive and hence lies along the nonnegative real frequency axis of the complex \( \omega \) plane.

The complex plane behavior at the simple pole singularity is then given by

\[
\phi(\omega_p, \theta) = -\omega_p n_p(\omega_p) + i \omega_p [n_1(\omega_p) - \theta],
\]

so that

\[
X(\omega_p) = -\omega_p n_p(\omega_p), \quad Y(\omega_p, 0) = \omega_p [n_1(\omega_p) - \theta],
\]

where \( n_p(\omega_p) \) is the real part of and \( n_1(\omega_p) \) is the imaginary part of the complex index of refraction along the nonnegative real frequency axis.

The saddle points \( \text{SP}_D^- \) and \( \text{SP}_N^- \) that are located in the left half of the complex \( \omega \) plane clearly do not interact with
of the path of steepest descent through the upper near saddle point SP, is equal to 0 as \( \theta \) increases from unity to \( \theta_1 \), \( \Delta \theta \) = \( \pi/6 \) at \( \theta = \theta_1 \), and the angle of slope of the path of steepest descent through the near saddle point SP\(_N^+\) is equal to \( \pi/4 \) for \( \theta > \theta_1 \). Notice that, although the value of \( \Delta \theta \) changes abruptly at \( \theta = \theta_1 \), the path of steepest descent varies in a continuous fashion with \( \theta \) for all \( \theta \geq 1 \). Substitution of these results into Eq. (C12) then yields

\[
\arg[\Delta(\theta)] = -\pi/2, \quad 1 < \theta < \theta_1,
\]

\[
\arg[\Delta(\theta)] = 0, \quad \theta = \theta_1,
\]

\[
\arg[\Delta(\theta)] = 3\pi/4, \quad \theta > \theta_1,
\]

where \( \theta_1 \) is as defined in Appendix C. Notice that \( \theta_0 > \theta_1 \) for all values of \( \omega_p \) in the domain \( 0 < \omega_p \leq (\omega_0^2 - \delta^2)^{1/2} \), while \( \theta_1 = \theta_0 \) for \( \omega_p = 0 \).

The uniform asymptotic contribution of the simple pole singularity at \( \omega = \omega_p \) with \( 0 < \omega_p \leq (\omega_0^2 - \delta^2)^{1/2} \) is then obtained from Eqs. (2.24) and (5.3) as follows.\(^4\) For \( \theta < \theta_0 \), \( \Im[\Delta(\theta)] < 0 \), and thus Eqs. (2.24) and (5.3a) yield

\[
A_\omega(z, t) \sim \frac{1}{2\pi} \Re \left[ i\gamma \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \frac{1}{\Delta(\theta)} \left( \frac{\pi c}{z} \right)^{1/2} \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] \theta < \theta_0,
\]

as \( z \to \infty \), where \( \omega_{SP} \) denotes \( \omega_{SP1} \) and \( \Delta(\theta) \) is given by Eq. (5.4a) for \( 1 < \theta < \theta_1 \), \( \omega_{SP} \) denotes \( \omega_{SPN} \) and \( \Delta(\theta) \) is given by Eq. (5.4b) at \( \theta = \theta_1 \), and \( \omega_{SP} \) denotes \( \omega_{SPN^+} \) and \( \Delta(\theta) \) is given by Eq. (5.4c) for \( \theta > \theta_1 \). At \( \theta = \theta_0 \), \( \Im[\Delta(\theta)] = 0 \), and Eqs. (2.24) and (5.3b) yield, for \( \omega_p \neq 0 \),

\[
A_\omega(z, t) \sim \frac{1}{2\pi} \Re \left[ i\gamma \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \frac{1}{\Delta(\theta)} \left( \frac{\pi c}{z} \right)^{1/2} \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \Re \left( \gamma \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right),
\]

\[
\theta = \theta_0, \quad \omega_p \neq 0
\]

(5.7)

as \( z \to \infty \), where \( \omega_{SP} \) denotes \( \omega_{SP1} \) and \( \Delta(\theta) \) is given by Eq. (5.4a) for \( \theta_0 < \theta < \theta_1 \), \( \omega_{SP} \) denotes \( \omega_{SPN} \) and \( \Delta(\theta) \) is given by Eq. (5.4b) at \( \theta = \theta_1 \), and \( \omega_{SP} \) denotes \( \omega_{SPN^+} \) and \( \Delta(\theta) \) is given by Eq. (5.4c) for \( \theta > \theta_1 \). For \( \theta > \theta_0 \), \( \Im[\Delta(\theta)] > 0 \), and thus Eqs. (2.24) and (5.3a) yield

\[
A_\omega(z, t) \sim \frac{1}{2\pi} \Re \left[ i\gamma \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \frac{1}{\Delta(\theta)} \left( \frac{\pi c}{z} \right)^{1/2} \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \Re \left( \gamma \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right),
\]

\[
\theta > \theta_1
\]

(5.8)

as \( z \to \infty \), where \( \omega_{SP} \) denotes \( \omega_{SP1} \) and \( \Delta(\theta) \) is given by Eq. (5.4a) for \( \theta_0 < \theta < \theta_1 \), \( \omega_{SP} \) denotes \( \omega_{SPN} \) and \( \Delta(\theta) \) is given by Eq. (5.4b) at \( \theta = \theta_1 \), and \( \omega_{SP} \) denotes \( \omega_{SPN^+} \) and \( \Delta(\theta) \) is given by Eq. (5.4c) for \( \theta > \theta_1 \).

For the special case in which \( \omega_p = 0 \), the upper near saddle point SP1 and the simple pole singularity coalesce when \( \theta = \theta_0 \), so that \( \theta_1 = \theta_0 \) and \( \Im[\Delta(\theta)] = 0 \). For \( \theta < \theta_0 \), \( \Im[\Delta(\theta)] < 0 \), and
Fig. 6. Interaction of the near saddle point with a simple pole singularity at \( \omega = \omega_p > 0 \). The shaded area indicates the region of the complex \( \omega \) plane wherein \( X_{\omega_p}(\theta) > X(\omega, \theta) \), where \( \omega_{SP} = \omega_{SP}^p \) for \( 1 < \theta < \theta_1 \), \( \omega_{SP} = \omega_{SP}^p \) at \( \theta = \theta_1 \), or \( \omega_{SP}^p \) for \( \theta > \theta_1 \).

thus the uniform asymptotic approximation of the pole contribution is given by relation (5.6), where \( \omega_{SP} \) denotes \( \omega_{SP}^p \) and \( \Delta(\theta) \) is given by Eq. (5.4a). At \( \theta = \theta_1 = \theta_0 \), \( \Delta(\theta_0) = 0 \), and Eq. (5.3c) applies for \( C(z, t) \) [cf. Eq. (2.14)]. Since the path \( P(\theta_0) \) crosses over the pole at \( \omega = \omega_p = 0 \), then

\[
\Delta(\theta_0) = \frac{1}{2 \pi} i \gamma \exp \left[ \frac{z}{c} \phi(0, \theta_0) \right].
\]

Since

\[
\phi(\omega_{SP}, \theta_0) = \phi(0, \theta_0) = 0,
\]

\[
\phi^{(2)}(\omega_{SP}, \theta_0) \approx -\frac{2 \beta \theta_0}{\theta_0^4},
\]

\[
\phi^{(3)}(\omega_{SP}, \theta_0) \approx -\frac{3 \alpha \beta^2}{\theta_0^4},
\]

then Eqs. (2.24a) and (5.3c) yield

\[
A(x, t_0) \sim \frac{\omega_0^2}{2b} \frac{2}{\pi \delta \omega_p} \left[ -\frac{1}{\omega_{SP}(\theta_0)} + \frac{i \alpha \gamma}{4 \beta} \right] Re \left[ \Gamma \left( -\frac{1}{\omega_{SP}(\theta_0)} + \frac{i \alpha \gamma}{4 \beta} \right) \right] + \frac{1}{2} Re(\gamma), \quad \theta = \theta_1, \quad \omega_p = 0 \quad (5.9)
\]

as \( z \to \infty \) with \( t_0 = (z/c)\theta_0 \). Notice that, even though the term \( 1/\omega_{SP}(\theta_0) \) is singular, since \( \omega_{SP}(\theta_0) = 0 \), this expression for \( A(x, t_0) \), when combined with the asymptotic behavior of the second precursor field, yields a uniform asymptotic approximation of the total field \( A(z, t) \) that is well behaved at \( \theta = \theta_0 \). Finally, for \( \theta > \theta_0 \), \( \text{Im}[\Delta(\theta)] > 0 \), and the uniform asymptotic approximation of the pole contribution is given by Eq. (5.8), where \( \omega_{SP} \) denotes \( \omega_{SP}^p \) and \( \Delta(\theta) \) is given by Eq. (5.4a) for \( \theta_1 < \theta < \theta_1 \), \( \omega_{SP} \) denotes \( \omega_{SP}^p \) and \( \Delta(\theta) \) is given by Eq. (5.4b) at \( \theta = \theta_1 \), and \( \omega_{SP} \) denotes \( \omega_{SP}^p \) and \( \Delta(\theta) \) is given by Eq. (5.4c) for \( \theta > \theta_1 \).

Taken together, the expressions in relations (5.6)-(5.8) constitute the uniform asymptotic approximation of the pole contribution at \( \omega = \omega_p \) with \( 0 < \omega_p \leq (\omega_0^2 - \delta^2)^{1/2} \), whereas the expressions in relations (5.6), (5.9), and (5.8) constitute the uniform asymptotic approximation of the pole contribution at \( \omega = \omega_p = 0 \). For values of \( \theta < \theta_0 \) and sufficiently large observation distances \( z \) such that the quantity \( |\Delta(\theta)(z/c)|^{1/2} \) is large, the dominant term in the asymptotic expansion of the complementary error function may be substituted into the asymptotic expression in relation (5.6) with the result that the first and second terms in that relation identically cancel. Hence, for values of \( \theta \) sufficiently less than \( \theta_0 \), there is no contribution to the asymptotic behavior of the total field from the simple pole singularity. For values of \( \theta > \theta_0 \) and sufficiently large observation distances \( z \) such that the quantity \( |\Delta(\theta)(z/c)|^{1/2} \) is large, the dominant term in the asymptotic expansion of \( \text{erfc}[-i\Delta(\theta)(z/c)^{1/2}] \) may be substituted into the asymptotic expression given in relation (5.8) with the result that
\[ A_c(z, t) \sim \text{Re} \left\{ \gamma \exp \left[ i \frac{z}{c} \phi(\omega_p, \theta) \right] \right\} \]

\[ = \exp \left\{ -\frac{i\alpha(\omega_p)}{\gamma} \cos[k(\omega_p)z - \omega_p t] \right\} \]

\[ - \gamma'' \sin[k(\omega_p)z - \omega_p t] \]

(5.10)

as \( z \to \infty \) with \( \theta \) bounded away from \( \theta_c \). This result is in agreement with the nonuniform approximation given in Ref. 5, in which \( \gamma' = \text{Re} (\gamma) \) and \( \gamma'' = \text{Im} (\gamma) \). The amplitude attenuation coefficient \( \alpha(\omega_p) \) is given by

\[ \alpha(\omega_p) = -\frac{1}{c} X(\omega_p) = \frac{\omega_p}{c} n_1(\omega_p), \]

(5.11)

and

\[ k(\omega_p) = \frac{\omega_p}{c} n_2(\omega_p) \]

(5.12)

is the propagation factor at the real frequency \( \omega_p \) in the dispersive medium.

For the input unit-step-function-modulated signal the initial envelope spectrum \( \tilde{u}(\omega - \omega_c) \) possesses a single simple pole singularity at the applied signal frequency \( \omega_c \) so that \( \omega_p = \omega_c \) and

\[ \gamma = \lim_{\omega \to \omega_c} \left[ (\omega - \omega_c) \frac{i}{\omega - \omega_p} \right] = i. \]

(5.13)

Substitution of this result into the asymptotic expressions given in relations (5.6)–(5.9) then yields the following set of equations describing the uniform asymptotic behavior of the pole contribution for applied signal frequencies \( \omega_c \) in the domain \( 0 \leq \omega_c \leq (\omega_c^2 - \delta^2)^{1/2} / 2 \):

\[ A_c(z, t) \sim \frac{1}{\pi} \text{Re} \left\{ i \frac{\pi}{\omega_c} \left[ \frac{\pi}{z} \right]^{1/2} \exp \left[ \frac{\pi}{c} \phi(\omega_c, \theta) \right] \right\} \]

\[ - \frac{1}{\Delta(\theta)} \left[ \frac{\pi}{z} \right]^{1/2} \exp \left[ \frac{\pi}{c} \phi(\omega_{SPD}, \theta) \right], \quad \theta < \theta_c \]

(5.14a)

\[ A_c(z, t_s) \sim \frac{1}{2\pi} \text{Re} \left\{ 2 \left[ \frac{\pi}{z} \right]^{1/2} \exp \left[ \frac{x}{c} \phi(\omega_{SPD}, \theta_s) \right] \right\} \]

\[ - \frac{1}{\Delta(\theta)} \left[ \frac{\pi}{z} \right]^{1/2} \exp \left[ \frac{\pi}{c} \phi(\omega_{SPD}, \theta_s) \right] \]

\[ - (1/2) \exp \left\{ -i\alpha(\omega_c) \sin[h(\omega_c z - \omega_c t_s)] \right\}, \]

\[ \theta = \theta_s = \text{ct}/z, \quad \omega_c \neq 0, \]

(5.14b)

\[ A_c(z, t_s) \sim \frac{\omega_o^2}{2b} \left[ \frac{\theta_c}{\pi b z} \right]^{1/2} \exp \left[ \frac{x}{c} \phi(\omega_c, \theta_s) \right], \quad \theta = \theta_s, \quad \omega_c = 0, \]

(5.14c)

\[ A_c(z, t_s) \sim \frac{1}{2\pi} \text{Re} \left\{ -i \frac{\pi}{\omega_c} \left[ \frac{\pi}{z} \right]^{1/2} \exp \left[ \frac{\pi}{c} \phi(\omega_c, \theta_s) \right] \right\} \]

\[ - \frac{1}{\Delta(\theta)} \left[ \frac{\pi}{z} \right]^{1/2} \exp \left[ \frac{\pi}{c} \phi(\omega_{SPD}, \theta_s) \right] \]

\[ - \exp \left\{ -i\alpha(\omega_c) \sin[h(\omega_c z - \omega_c t)] \right\}, \quad \theta < \theta_s \]

(5.14d)

as \( z \to \infty \), where

\[ F(t) = \exp(-t^2) \int_0^t \exp(t^2) dt \]

is Dawson's integral.

The dynamic behavior of the pole contribution as described by the asymptotic expressions given in relations (5.14a), (5.14b), and (5.14d) for the unit-step-function-modulated signal is illustrated in Fig. 7 as a function of the space–time parameter \( \theta \) at the fixed propagation distance \( z = 1 \times 10^{-3} \text{ cm} \) for a fixed carrier frequency \( \omega_c = 1 \times 10^{16} \text{ sec}^{-1} \). For this particular case the maximum amplitude attained by the field contribution \( A_c(z, t) \) from the simple pole is an order of magnitude larger than that of the first precursor field and an order of magnitude less than that of the second precursor field (see Fig. 5). Notice that both the amplitude and the oscillation frequency of \( A_c(z, t) \) settle down to their appropriate steady-state values for \( \theta > \theta_c > \theta_o \). Furthermore, for all \( \theta > \theta_c > \theta_o \), the amplitude of the second precursor field is less than that of \( A_c(z, t) \) and is negligible for values of \( \theta \) only slightly greater than \( \theta_c \) for the particular case illustrated here. The specific value of \( \theta_c \) is defined in Subsection 6.C.

B. Frequencies \( \omega_p \) in the Domain \( \omega_p \geq (\omega_c^2 - \delta^2)^{1/2} \)

For finite real frequencies \( \omega_p \) in the domain \( \omega_p \geq (\omega_c^2 - \delta^2)^{1/2} \), which is above the medium absorption band, it is the distant saddle point SPD+ in the right half of the complex \( \omega \) plane that interacts with the sample pole singularity at \( \omega_c = \omega_c \). The results of Eqs. (2.24) and (5.3) then apply for \( \omega_{SPD} \) in the right half of the complex \( \omega \) plane that interacts with the sample pole singularity at \( \omega = \omega_c \).

The results of Eqs. (2.24) and (5.3) then apply for \( \omega_{SPD} \) in the right half of the complex \( \omega \) plane that interacts with the sample pole singularity at \( \omega = \omega_c \).

Further-
3π/4 for all θ ≥ 1. Furthermore, the angle of slope δc of the vector from osp+ to ωp is seen to decrease monotonically with increasing θ ≥ 1 and lies within the domain π ≥ α ≥ 0. It then follows from Eq. (C12) with n = 0 that
\[
\frac{\pi}{4} \geq \arg(\Delta(\theta)) > 0, \quad 1 \leq \theta < \theta_s, 
\]
\[
\arg(\Delta(\theta)) = 0, \quad \theta = \theta_s, 
\]
\[
0 > \arg(\Delta(\theta)) > -\frac{3\pi}{4}, \quad \theta > \theta_s, 
\]
where \(\arg(\Delta(\theta))\) decreases monotonically with increasing θ.

The uniform asymptotic approximation of the contribution from the simple pole singularity at \(\omega = \omega_p \geq (\omega_0^2 - \delta_0^2)^{1/2}\) is then as follows.  For \(1 \leq \theta < \theta_s\), \(\mathrm{Im}[\Delta(\theta)] > 0\), and Eqs. (5.16a) and (5.1a) yield
\[
A_c(z, t) \sim \frac{1}{2\pi} \Re \left[ i\gamma \left[ i\pi \operatorname{erfc} \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \frac{1}{\Delta(\theta)} \left( \frac{\pi c}{z} \right)^{1/2} \exp \left( \frac{z}{c} \phi(\omega_{SP}, \theta) \right) \right] 
\]
as \(z \to \infty\). For \(1 \leq \theta < \theta_s\), \(\mathrm{Im}[\Delta(\theta)] > 0\), and Eqs. (5.16a) and (5.3a) yield
\[
A_c(z, t) \sim \frac{1}{2\pi} \Re \left[ i\gamma \left[ i\pi \operatorname{erfc} \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \frac{1}{\Delta(\theta)} \left( \frac{\pi c}{z} \right)^{1/2} \exp \left( \frac{z}{c} \phi(\omega_{SP}, \theta) \right) \right] + \Re \left[ \gamma \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] 
\]
as \(z \to \infty\). Finally, for \(\theta > \theta_s\), \(\mathrm{Im}[\Delta(\theta)] < 0\), and Eqs. (5.16a) and (5.3a) yield
\[
A_c(z, t) \sim \frac{1}{2\pi} \Re \left[ i\gamma \left[ i\pi \operatorname{erfc} \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \frac{1}{\Delta(\theta)} \left( \frac{\pi c}{z} \right)^{1/2} \exp \left( \frac{z}{c} \phi(\omega_{SP}, \theta) \right) \right] + \Re \left[ \gamma \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] 
\]
as \(z \to \infty\).

Taken together, the expressions in relations (5.17)–(5.19) constitute the uniform asymptotic approximation of the pole contribution at \(\omega = \omega_p \geq (\omega_0^2 - \delta_0^2)^{1/2}\). For fixed values of \(\theta < \theta_s\) and sufficiently large values of the propagation distance \(z\) such that the quantity \(|\Delta(\theta)|(z/c)^{1/2}\) is large, the dominant term in the asymptotic expansion of \(\operatorname{erfc}[-i\Delta(\theta)(z/c)^{1/2}]\) may be substituted into the asymptotic expression in relation (5.17) with the result that the first and second terms in that relation identically cancel. Hence, for values of \(\theta\) sufficiently less than \(\theta_s\), there is no contribution to the asymptotic behavior of the total field from the simple pole singularity. For fixed values of \(\theta > \theta_s\) and values of the observation distance \(z\) sufficiently large that the quantity \(|\Delta(\theta)|(z/c)^{1/2}\) is large, the dominant term in the asymptotic expansion of \(\operatorname{erfc}[-i\Delta(\theta)(z/c)^{1/2}]\) may be substituted into the asymptotic expression in relation (5.19) with the result that
\[
A_c(z, t) \sim \exp[-z\alpha(\omega_p)] \left[ \gamma' \cos[k(\omega_p)z - \omega_c t] - \gamma'' \sin[k(\omega_p)z - \omega_c t] \right] 
\]
as \(z \to \infty\), where \(\gamma'\) and \(\gamma''\) are defined in relations (5.13). For the input-step-function-modulated signal with applied signal frequency \(\omega_c \geq (\omega_0^2 - \delta_0^2)^{1/2}\) the asymptotic expressions given in relations (5.17)–(5.19) become, with substitution of Eq. (5.13),
\[
A_c(z, t) \sim \frac{1}{2\pi} \Re \left[ -i\pi \operatorname{erfc} \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] 
\]
for \(1 \leq \theta < \theta_s\), \(\mathrm{Im}[\Delta(\theta)] > 0\), and Eqs. (5.16a) and (5.3a) yield
\[
A_c(z, t) \sim \frac{1}{2\pi} \Re \left[ -i\pi \operatorname{erfc} \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_{SP}, \theta) \right) \right] 
\]
for \(1 \leq \theta < \theta_s\), \(\mathrm{Im}[\Delta(\theta)] > 0\), and Eqs. (5.16a) and (5.3a) yield
\[
A_c(z, t) \sim \frac{1}{2\pi} \Re \left[ -i\gamma \left[ i\pi \operatorname{erfc} \left( -i\Delta(\theta) \left( \frac{z}{c} \right)^{1/2} \right) \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] + \frac{1}{\Delta(\theta)} \left( \frac{\pi c}{z} \right)^{1/2} \exp \left( \frac{z}{c} \phi(\omega_{SP}, \theta) \right) \right] + \Re \left[ \gamma \exp \left( \frac{z}{c} \phi(\omega_p, \theta) \right) \right] 
\]
as \(z \to \infty\), where \(P(\xi)\) again denotes Dawson’s integral.
C. Frequencies $\omega_p$ in the Domain $(\omega_0^2 - i\Delta)^{1/2} < \omega_p < (\omega_0^2 - i\Delta)^{1/2}$

For values of $\omega_p$ in the domain $(\omega_0^2 - i\Delta)^{1/2} < \omega_p < (\omega_0^2 - i\Delta)^{1/2}$, which is within the absorption band of the single-resonance Lorentz medium, neither the near nor the distant saddle point comes within close proximity of the simple pole singularity located at $\omega = \omega_p$. In that case the quantity $|\Delta(\theta)(z/c)|^{1/2}$ is large for all $\theta \geq 1$, and the dominant term in the corresponding asymptotic expansion of the complementary error function can be substituted into Eqs. (C5) with the result that obtained by Brillouin because $\omega_p$ is asymptotically negligible in comparison with the second precursor field $A_p(z, t)$ is predominant, and $A_s(z, t)$ is asymptotically negligible when the pole contribution $A_s(z, t)$ is predominant. Two or three terms become important at the same time during periods of transition, giving a continuous asymptotic description of the time evolution of the total field $A(z, t)$ for large $z$ for all $\theta \geq 1$.

For the input unit-step-function-modulated signal with applied signal frequency $\omega_c$ in the absorption band, the above results become

$$A_s(z, t) \sim \frac{1}{2} \text{Re} \left\{ \gamma \exp \left[ \frac{z}{c} \phi(\omega_p, \theta) \right] \right\}$$

$$= \left( \frac{1}{2} \right) \exp[-2\pi\omega_c|\theta'| \exp[k(\omega_c)z - \omega_p t]]$$

$$- \gamma' \sin[k(\omega_c)z - \omega_p t], \quad \theta = \theta_s = ct / z$$

(5.22)

as $z \to \infty$; and for $\theta > \theta_s$

$$A_s(z, t) \sim \text{Re} \left\{ \gamma \exp \left[ \frac{z}{c} \phi(\omega_p, \theta) \right] \right\}$$

$$= \exp[-2\pi\omega_c|\theta'| \exp[k(\omega_c)z - \omega_p t]]$$

$$- \gamma' \sin[k(\omega_c)z - \omega_p t], \quad \theta > \theta_s$$

(5.23)

as $z \to \infty$. These results are the same as those given in Ref. 5 by the nonuniform asymptotic approximation.

For the input unit-step-function-modulated signal with applied signal frequency $\omega_c$ in the absorption band, the above results become

$$A_s(z, t) = 0, \quad \theta < \theta_s$$

(5.24a)

$$A_s(z, t) \sim -(1/2) \exp[-2\pi\omega_c|\theta'| \exp[k(\omega_c)z - \omega_p t]], \quad \theta = \theta_s = ct / z$$

(5.24b)

$$A_s(z, t) \sim - \exp[-2\pi\omega_c|\theta'| \exp[k(\omega_c)z - \omega_p t]], \quad \theta > \theta_s$$

(5.24c)

as $z \to \infty$. These equations are the same as the expressions that Brillouin obtained for the simple pole contribution for all positive values of the carrier frequency $\omega_c$. In the limit of large values of the quantity $|\Delta(\theta)(z/c)|^{1/2}$ relations (5.14) and (5.21) also reduce to the same expressions. Even the value of the space–time parameter $\theta$ at which the discontinuous jump in the behavior of $A_s(z, t)$ occurs is the same as that obtained by Brillouin because $\theta_s$ is taken in the uniform asymptotic analysis presented here to be the value of $\theta$ at which the path of steepest descent crosses the simple pole singularity located at $\omega_p = \omega_c$. The nonuniform asymptotic analysis presented by Oughstun and Sherman yielded the same expressions for $A_s(z, t)$ as were obtained by Brillouin, but the value of $\theta_s$ can be different since $\theta_s$ is then the value of $\theta$ at which an arbitrary Olver-type path crosses the pole. The difference in values obtained for $\theta_s$ is of no consequence, however, since $A_s(z, t)$ is asymptotically negligible in the final expression for the total field $A(z, t)$ for values of $\theta$ in a range that includes all possible values of $\theta_p$. Although Brillouin associated $\theta_p$ with the time of arrival of the main signal, that interpretation was shown to be incorrect by Baerwald, who obtained a uniform asymptotic approximation for the interaction of a saddle point with a nearby pole singularity and showed that the signal velocity reached a minimum near the medium resonance frequency. Similar results were obtained by Trizna and Weber for the dilute-medium case. However, the complete asymptotic description of a signal arrival and resultant signal velocity in the mature dispersion regime without any additional approximations imposed on the medium properties was obtained only recently by Oughstun and Sherman, using the nonuniform theory based on Olver’s method, and is confirmed here with the complete uniform asymptotic theory.

6. CONTINUOUS EVOLUTION OF THE TOTAL FIELD

In this final section the results of the previous sections are combined to obtain the uniform asymptotic evolution of the propagated field $A(z, t)$, which is expressed in the form [cf. Eq. (2.23)]

$$A(z, t) = A_p(z, t) + A_B(z, t) + A_s(z, t)$$

(6.1)

for $\theta = ct / z \geq 1$.

A. Total Precursor Field and Uniform Asymptotic Behavior of the Delta-Function Pulse

The total precursor field $A_p(z, t)$ is the combined contribution to $A(z, t)$ from both the near and the distant saddle points for $\theta \geq 1$, that is, the sum of the Sommerfeld and Brillouin precursor fields (for a single-resonance Lorentz medium),

$$A_p(z, t) = A_s(z, t) + A_B(z, t)$$

(6.2)

Since there is no pole contribution for the input delta-function pulse, this equation yields the uniform asymptotic description of the total field evolution in that case, where $A_s(z, t)$ is given by the asymptotic expression in relation (3.7) and $A_B(z, t)$ is given by the asymptotic expressions in relations (4.29)–(4.33).

For sufficiently large values of the propagation distance $z$ and for values of $\theta \geq 1$ bounded away from $\theta_B$ [cf. Eq. (2.25)], the second precursor $A_B(z, t)$ is asymptotically negligible in comparison with the first precursor field $A_s(z, t)$ when $1 \leq \theta < \theta_B$, and the first precursor field $A_s(z, t)$ is asymptotically negligible in comparison with the second precursor field $A_p(z, t)$ when $\theta > \theta_B$. Both $A_s(z, t)$ and $A_p(z, t)$ are important in the transition region between the first and second precursor fields in a neighborhood of $\theta = \theta_B$. When asymptotic approximations of $A_s(z, t)$ and $A_B(z, t)$, uniformly valid for $\theta$ in a specific domain, are applied in Eq. (6.2), it follows from the results obtained in Appendix D that the result is an asymptotic approximation of $A(z, t)$ that is uniformly valid over the same domain of values of $\theta$. Hence substitution into Eq. (6.2) of the uniform asymptotic ap-
proximations of \( A_c(z, t) \) and \( A_p(z, t) \) that were obtained in Sections 3 and 4 provides an asymptotic approximation of the total precursor field \( A_p(z, t) \), as well as the total field for the input delta-function pulse, that is uniformly valid for all \( \theta \geq 1 \).

B. Resonance Peaks of the Precursors and the Main Signal

Examination of the expressions for the Sommerfeld and Brillouin precursor fields shows that each of them exhibits a resonance peak as \( \theta \) varies if the relevant saddle point passes near a first-order pole of the spectral function. Indeed, it is readily apparent from the general expression for the asymptotic contribution of a first-order saddle point, as given by the first term in relation (C5), viz.,

\[
I(z, t) \sim q(\omega_p) \left[ -\frac{2\pi}{zp^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)],
\]

that \( I(z, t) \) becomes large if \( \omega_p(\theta) \) approaches a pole \( \omega_p \) of \( q(\omega) \) as \( \theta \) varies.

Since the condition that leads to the resonance peak in the saddle-point contribution is that the saddle point passes near the first-order pole, it is then necessary that the uniform asymptotic expression for the pole contribution \( A_c(z, t) \) be included in the consideration of this phenomenon. The asymptotic approximation of the integral representation of \( A(z, t) \) can then be written as

\[
A(z, t) \sim q(\omega_p) \left[ -\frac{2\pi}{zp^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)] + A_c(z, t).
\]

From relation (C5), if the simple pole and the saddle point do not coalesce, the uniform asymptotic approximation of \( A_c(z, t) \) can be written as

\[
A_c(z, t) \sim q(\omega_p) \left[ -\frac{2\pi}{zp^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)] + f_3(\omega_p),
\]

where \( f_3(\omega_p) \) is an analytic function of \( \omega_p \) and \( \gamma \) is the residue of the pole of the spectral amplitude function \( q(\omega) \). Since \( \omega_p \) is a first-order saddle point of \( p(\omega, \theta) \), the phase function evaluation at \( \omega = \omega_p \) can be expanded in a Taylor series about \( \omega_p \) in the form

\[
p(\omega_p, \theta) = p(\omega_p, \theta) + (1/2)p^{(2)}(\omega_p, \theta)(\omega_p - \omega_p)^2 + \ldots
\]

for \( \omega_p \) sufficiently close to \( \omega_p \). The quantity \( \Delta(\theta) \) defined in Eq. (C7) may then be written as

\[
\Delta(\theta) = \left[ -\frac{1}{2} p^{(2)}(\omega_p, \theta) \right]^{1/2}(\omega_p - \omega_p) + f_1(\omega_p),
\]

where \( f_1(\omega_p) \) is an analytic function of \( \omega_p \) that goes to zero as \( \omega_p \) approaches \( \omega_p \). As a result, relation (6.5) can be expressed as

\[
A_c(z, t) \sim \frac{\gamma}{[-(1/2)p^{(2)}(\omega_p, \theta)]^{1/2}(\omega_p - \omega_p)} \times \left[ \frac{\pi}{2} \right]^{1/2} \exp[zp(\omega_p, \theta)] + f_3(\omega_p),
\]

where \( f_3(\omega_p) \) is an analytic function of \( \omega_p \)

Similarly, since \( q(\omega) \) has a first-order pole at \( \omega = \omega_p \) with residue \( \gamma \), the first term appearing in relation (6.4) can be written as

\[
q(\omega_p) \left[ 1 - \frac{2\pi}{zp^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)]
\]

\[
= \frac{\gamma}{\omega_p - \omega_p} \left[ 1 - \frac{2\pi}{zp^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)] + f_3(\omega_p),
\]

where \( f_3(\omega_p) \) is an analytic function of \( \omega_p \). Substitution of relations (6.8) and (6.9) into relation (6.4) then yields

\[
A(z, t) \sim A_p(\omega_p) + f_3(\omega_p).
\]

Hence \( A(z, t) \) is an analytic function of \( \omega_p \) in a neighborhood of the saddle point \( \omega_p \) and therefore cannot have a singularity at \( \omega_p = \omega_p \). Consequently, the resonance peak of the precursor field is canceled exactly by an identical resonance peak of the pole contribution. The total propagated field \( A(z, t) \) then does not exhibit the resonance peaks exhibited by its component subfields \( A_p(z, t) \) and \( A_c(z, t) \), provided that the uniform asymptotic description of the pole contribution is used.

A resonance peak is not observed in the Brillouin precursor field for the unit-step-function-modulated signal depicted in Fig. 5 because the medium damping constant \( \delta = 0.28 \times 10^{10}/\text{sec} \) is too large (the saddle point and the pole are then sufficiently well separated). An example of this resonance peak phenomenon in the component fields for the unit-step-function-modulated signal in a weakly dispersive medium with parameters

\[
\omega_0 = 2.0 \times 10^{10}/\text{sec},
\]

\[
b^2 = 0.4 \times 10^{10}/\text{sec},
\]

\[
\delta = 1.7 \times 10^{12}/\text{sec}
\]

is illustrated in Fig. 9. The applied signal frequency is \( \omega_c = 0.1 \times 10^{10}/\text{sec} \), which is below the absorption band of the medium, so that the resonance peak phenomenon occurs in the Brillouin precursor. Notice that there is no resonance phenomenon in the total field evolution, shown in Fig. 9(c), owing to the cancellation between the resonance peak in the Brillouin precursor, shown in Fig. 9(a), and the resonance peak in the pole contribution, shown in Fig. 9(b), when these two component fields are added together to give \( A(z, t) \). (The amplitude of the first precursor field in this range of values of \( \theta \) is less than \( 10^{-72} \) so that it is asymptotically negligible over this \( \theta \) interval.) In the nonuniform asymptotic theory presented in Refs. 1–5 the resonance peak appears in the appropriate precursor field but does not appear in the pole contribution, so that it remains erroneously in the total field evolution.

C. Signal Arrival and Uniform Asymptotic Behavior of the Unit-Step-Function-Modulated Signal

From the results of the uniform asymptotic description of the pole contribution given in Section 5, the contribution of the simple pole singularity at \( \omega_p \) is seen to occur when the original contour of integration, which extends along the straight line from \( ia - \infty \) to \( ia + \infty \) in the upper half of the complex \( \omega \) plane, lies on the side of the pole singularity opposite that of the Oliver-type path \( P(\theta) \) through the relevant saddle points; that is, \( P(\theta) \) and the original contour of
integration lie on the same side of the pole for \( \theta < \theta_s \) and on opposite sides for \( \theta > \theta_s \), as stated in Eqs. (C3). Consequently, for \( \theta < \theta_s \), the pole is not crossed when the original contour is deformed to \( P(\theta) \), and there is no residue contribution, whereas for \( \theta > \theta_s \), the pole is crossed in deforming the original contour of integration to \( P(\theta) \), and there is no residue contribution, to form the total field \( A(z, t) \).

For \( \theta > \theta_s \) in the domain \( 0 < \omega_p \leq (\omega_s^2 - \delta)^{1/2} \) the uniform asymptotic approximation of the pole contribution at \( \theta = \theta_s \) is given by

\[
A_c(z, t) \sim \exp[-z_0(\omega_p)]\gamma_p' \cos[k(\omega_p)z - \omega_p t] \\
- \gamma_p'' \sin[k(\omega_p)z - \omega_p t]
\]

(6.13)

for \( \theta > \theta_s \), as \( z \to \infty \), when \( \omega_p \) is real and positive. Here \( \gamma_p' = \text{Re}(\gamma_p) \) and \( \gamma_p'' = \text{Im}(\gamma_p) \), where

\[
\gamma_p = \lim_{\omega \to \omega_p} [(\omega - \omega_p)\tilde{A}(\omega - \omega_p)]
\]

(6.14)

is the residue of the simple pole singularity at \( \omega = \omega_p \) of the spectral amplitude function.

The pole contribution [relation (6.13)] is physically due to the frequency component at the real frequency \( \omega_p \). For the unit-step-function-modulated signal the pole occurs at \( \omega_p = \omega_s \), the applied signal frequency of the source. The pole contribution is then seen to correspond to the signal arrival in the dispersive medium. Since the signal arrival is defined completely by the value of \( \theta = \theta_s \) defined by Eq. (6.12), its frequency dependence, as well as the frequency dependence of the resultant signal velocity, is precisely the same as that given in the nonuniform approximation of Ref. 5.

The uniform asymptotic approximation of the pole contribution at \( \omega = \omega_p \) takes on a particularly useful form for \( \theta = \theta_s \). For example, for values of \( \omega_p \) in the domain \( 0 < \omega_p \leq (\omega_s^2 - \delta)^{1/2} \) the uniform asymptotic approximation of the pole contribution at \( \theta = \theta_s \) is given by relation (5.8), where \( \omega_p \) denotes \( \omega_{\text{SPM}} \) and \( \Delta(\theta) \) is given by Eq. (5.4c). At \( \theta = \theta_s \), \( \delta_{\text{SD}} = \pi/4 \), so that

\[
\arg \left[-i\Delta(\theta) \left(\frac{z}{c}\right)^{1/2}\right] = -\frac{\pi}{4}
\]

and the complementary error function appearing in relation (5.8) may be replaced by
provided a complete uniform asymptotic description of electro-

The uniform asymptotic description of each of the precursor fields and the pole contributions presented here provide a general, rigorous framework in which the analysis of a large class of input pulse forms may be based directly. From a practical, computational point of view, the uniform asymptotic descriptions of the Sommerfeld and Brillouin precursor fields are expressed in terms of special magnetic pulse propagation in a linear dispersive medium with absorption as described by the classical Lorentz model. Because the recently derived expressions\(^6\) for the dynamical saddle-point locations have been used, which are accurate approximations over the entire space–time domain of interest, the resultant asymptotic expressions provide a complete, uniformly valid description of the entire dynamical field evolution in the mature dispersion limit. A resonance peak, which may occur either in the Sommerfeld precursor field if the distant saddle point passes close by a first-order pole of the spectral amplitude function or in the Brillouin precursor field if the near saddle point passes close by the pole, in a weakly absorptive medium has been shown to be canceled by an identical resonance peak of the opposite sign that is present in the uniform asymptotic approximation of the pole contribution. This result is not obtained with the nonuniform asymptotic theory\(^1\) of resonance peaks in a highly absorptive and dispersive medium. The uniform asymptotic description of the total field evolution for an input unit-step-function-modulated signal with carrier frequency \(\omega_c = 1.0 \times 10^{16}/\text{sec}\) at a propagation distance of \(z = 1 \times 10^{-3}\) cm is illustrated in Fig. 10 for the case of Brillouin's choice of the Lorentz-medium parameters \((\omega_s = 4.0 \times 10^{16}/\text{sec}, \beta_s = 20 \times 10^{23}/\text{sec}^3, \delta = 0.28 \times 10^{16}/\text{sec})\), which is representative of a highly absorptive medium. The observation distance \(z\) at which the field \(A(z, t)\) has been calculated here from the uniform asymptotic expressions in relations (3.9), (4.36)–(4.40), and (5.14) is \(z = 1 \times 10^{-3}\) cm so that Fig. 5 corresponds to the contribution \(A_0(z, t)\) and Fig. 7 corresponds to the contribution \(A_s(z, t)\) to the total field \(A(z, t)\). The Sommerfeld precursor field \(A_s(z, t)\) is negligible over the range of values of \(\theta\) indicated in Fig. 10. Notice the continuous transition from a state in which the second precursor field is the dominant contribution to the total field \(A(z, t)\) over to a state in which the pole contribution is the dominant contribution to \(A(z, t)\), which occurs for values of \(\theta\) near \(\theta_c = 1.5875\).

7. DISCUSSION

This rather involved treatment has, for the first time, provided a complete uniform asymptotic description of electro-
functions (e.g., the Bessel functions and the Airy function, respectively) that are relatively easy to compute. Unfortunately, this is not so for the uniform asymptotic representation of the pole contribution, which is expressed in terms of the complementary error function of a complex argument. We are unaware of any numerical routine for calculating this complex-valued function with a known degree of accuracy over a large range in magnitude and phase of its complex argument, and we were forced to construct our own, which is based on a simple matched series and asymptotic expansion of the complementary error function.

Finally, and perhaps most importantly, the physical description of dispersive pulse dynamics afforded by the uniform asymptotic theory is, with the exception of the resonance peak phenomenon (which was conveniently ignored in the nonuniform theory), precisely the same as that obtained in the asymptotic theory is, with the exception of the resonance peak phenomenon (which was conveniently ignored in the nonuniform theory of Ref. 5. In particular, the signal has been shown here to arrive at the same value of the nonuniform case as in the nonuniform theory, so that the resultant signal velocity is precisely that given in Ref. 5, including the bifurcation of the signal velocity when \( \omega_c > \omega_{SB} \). The value of the uniform asymptotic theory presented here is simply that it provides a complete, valid description of the asymptotic field evolution over the entire space–time domain in the mature dispersion regime. Furthermore, it completely validates the results obtained in the nonuniform theory of Ref. 5. This final point is critical to the new physical description of dispersive pulse dynamics\(^{13}\) that replaces the old group-velocity description and reduces to it in the weak-absorption, weak-dispersion limit.

**APPENDIX A: UNIFORM ASYMPTOTIC EXPANSION FOR TWO FIRST-ORDER SADDLE POINTS AT INFINITY**

The uniform asymptotic expansion resulting from two first-order saddle points \( \omega_\pm(\theta) \) with equal imaginary parts and with real parts that approach \( \pm \infty \), respectively, as \( \theta \) approaches unity from above is provided by the following theorem by Handelsman and Bleistein\(^{12}\):

**Theorem (Handelsman and Bleistein):** In the integrand of the contour integral

\[
I(z, \theta) = \int_{P(\theta)} q(\omega) \exp(\xi \psi(\omega, \theta)) d\omega \tag{A1}
\]

with real \( z \), let the function \( p(\omega, \theta) \) possess two first-order saddle points \( \omega_\pm(\theta) \) with equal imaginary parts and whose real parts approach \( \pm \infty \), respectively, as the parameter \( \theta \) approaches unity from above. Let the contour \( P(\theta) = P^+(\theta) + P^-(\theta) \) be a continuous function of \( \theta \), where \( P^+(\theta) \) is a path of steepest descent through \( \omega_\pm(\theta) \) with one end point satisfying \( \Re(\omega) = \pm \infty \). Furthermore, let the function \( \psi(\omega, \theta) \) defined by

\[
\psi(\omega, \theta) = -i p(\omega, \theta) \tag{A2}
\]

possess a Laurent-series expansion of the form

\[
\psi(\omega, \theta) \equiv \omega(1 - \theta) + \sum_{n=0}^\infty a_n(\theta) \omega^{-n} \tag{A3}
\]

for all \( \omega \) such that \( |\omega| \geq R_1 \) and for all \( \theta \) in the closed interval \([1, \theta']\), where \( R_1 \) is some finite positive constant and \( \theta' > 1 \) is a positive constant. All other saddle points of \( p(\omega, \theta) \), if any, are assumed to be finite in number and confined to some bounded region in the complex \( \omega \) plane such that \( |\omega| \leq R_2 < R_1 \) for all \( \theta \in [1, \theta'] \). Moreover, the amplitude function \( q(\omega) \) may be written in the form

\[
q(\omega) = \omega^{-(1+i)(\theta)} q(\omega) \tag{A4}
\]

for large \( |\omega| \) with real \( \nu > 0 \), where the function \( q(\omega) \) has a Laurent-series expansion that is convergent for \( |\omega| \geq R_1 \) and is such that

\[
\lim_{|\omega| \to \infty} q(\omega) = 0. \tag{A5}
\]

Then, subject to these conditions, the integral \( I(z, \theta) \) satisfies

\[
I(z, \theta) = -2\pi i \exp(-iz\beta(\theta)) \left[ 2\alpha(\theta) \exp \left( -i \frac{\pi}{2} \right) \right] \times \{ \gamma_0 p(\omega(z)) + 2\alpha(\theta) \exp \left( -i \frac{\pi}{2} \right) \gamma_1 \} + R(z, \theta), \tag{A6}
\]

where (with \( K \) a positive real constant independent of both \( \theta \) and \( z \))

\[
|R(z, \theta)| \leq K \frac{|2\alpha(\theta)|^{1+|z|}}{z} \left[ |J_{\nu+1}(\alpha(z))| + |J_{\nu+2}(\alpha(z))| \right] \tag{A7}
\]

for \( z > 0 \) and \( \theta \in [1, \theta'] \). End of theorem.

The remainder term in relation (A7) is small for large \( z \) independent of \( \alpha(\theta) \). The coefficients appearing in Eq. (A6) and relation (A7) are given by

\[
\begin{align*}
\alpha(\theta) &= -(1/2)\{\psi(\omega_+, \theta) - \psi(\omega_-, \theta)\}, \tag{A8} \\
\beta(\theta) &= -(1/2)\{\psi(\omega_+, \theta) + \psi(\omega_-, \theta)\}, \tag{A9} \\
\gamma_0(\theta) &= \frac{1}{2} \left\{ q(\omega_+) \left[ \frac{1}{2\alpha(\theta)} \right] \right\}^{1+} - \frac{4\alpha^2(\theta)}{\psi^2(\omega_+, \theta)} \right\}^{1/2} \\
&\quad + q(\omega_-) \frac{1}{2\alpha(\theta)} \left\{ \frac{1}{\psi^2(\omega_-, \theta)} \right\}^{1/2}, \tag{A10} \\
\gamma_1(\theta) &= \frac{1}{4\alpha(\theta)} \left\{ q(\omega_+) \left[ \frac{1}{2\alpha(\theta)} \right] \right\}^{1+} - \frac{4\alpha^2(\theta)}{\psi^2(\omega_+, \theta)} \right\}^{1/2} \\
&\quad - q(\omega_-) \frac{1}{2\alpha(\theta)} \left\{ \frac{1}{\psi^2(\omega_-, \theta)} \right\}^{1/2}. \tag{A11}
\end{align*}
\]

The argument \( \tilde{\alpha}_\pm \) of the quantity \( \{\nu^2 \psi^2(\omega, \theta)\} \) is chosen so as to satisfy the inequality

\[
|\tilde{\alpha}_\pm + \theta + \nu \tilde{\alpha}_\pm'| \leq \pi/2, \tag{A12}
\]

with \( \theta = \arg(iz) = \pi/2 \) for real and positive \( z \) and where \( \tilde{\alpha}_\pm' \) is the angle of slope of the contour \( P^+(\theta) \) leading away from the saddle point \( \omega_\pm \) in the limit as \( \theta \to 1^+ \). For values of \( \theta \) close to unity these coefficients reduce to

\[
\begin{align*}
\alpha(\theta) &= -2(\pi \alpha(\theta)(\theta - 1)^{1/2} + \mathcal{O}[\theta - 1^{1/2}], \tag{A13} \\
\beta(\theta) &= \mathcal{O}[\theta - 1^{1/2}], \tag{A14}
\end{align*}
\]

and the coefficients \( \gamma_0(\theta) \) and \( \gamma_1(\theta) \) are both \( \mathcal{O}(1) \) for \( \theta \in [1, \theta'] \). The proper branch of \( \alpha^{1/2}(\theta) \) is chosen so as to satisfy
where the function \( f(r) \) is given by

\[
\frac{4\alpha^2(\theta)}{\psi^{(2)}(\omega_2, \theta)} \right)^{1/2} = 2^{1/2} \alpha_1(\theta)[1 + \mathcal{O}((1 - 1)^{1/2})] \tag{A15}
\]
as \( \theta \to 1 \) for all \( \theta \in [1, \theta'] \).

If \( \nu < 0 \) for Eq. (A4), the uniform asymptotic approximation provided by this theorem is still applicable for \( \theta \in [1, \theta'] \), provided that its limiting value as \( \theta \) approaches unity from above is finite.

**APPENDIX B: UNIFORM ASYMPTOTIC EXPANSION FOR TWO NEIGHBORING FIRST-ORDER SADDLE POINTS**

The uniform asymptotic expansion for two neighboring first-order saddle points \( \omega_1(\theta) \) and \( \omega_2(\theta) \) that coalesce into a single second-order saddle point \( \omega_0 \) when \( \theta = \theta_0 \), is provided by the following theorem of Chester et al.:

**Theorem (Chester, Friedman, and Ursell):** In the integrand of the contour integral

\[
I(z, \theta) = \int_{\Gamma(\theta)} q(\omega) \exp[z\phi(\omega, \theta)] d\omega \tag{B1}
\]

with real positive \( z \), let the functions \( p(\omega, \theta) \) and \( q(\omega) \) both be holomorphic in a domain containing the two first-order saddle points \( \omega_1(\theta) \) and \( \omega_2(\theta) \) of \( p(\omega, \theta) \), which vary in position in the complex \( \omega \) plane as the parameter \( \theta \) varies over a domain \( \mathcal{R} \) in such a way that as \( \theta \) approaches some critical value \( \theta_c \in \mathcal{R} \) the two saddle points coalesce into a single saddle point of second order; that is, when \( \theta \neq \theta_c \),

\[
\begin{align*}
\text{for } i = 1, 2: \quad &p^{(1)}(\omega_i) = p^{(1)}(\omega_i) = 0, \\
&\text{for } i = 1, 2: \quad &p^{(2)}(\omega_1) \neq 0, \\
&\text{for } i = 1, 2: \quad &p^{(2)}(\omega_2) \neq 0,
\end{align*}
\tag{B2}
\]

whereas, at \( \theta = \theta_c \), \( \omega_1 = \omega_2 = \omega_0 \)

\[
\begin{align*}
&\text{for } i = 1, 2: \quad &p^{(1)}(\omega_0) = p^{(2)}(\omega_0) = 0, \\
&\text{for } i = 1, 2: \quad &p^{(3)}(\omega_0) \neq 0.
\tag{B3}
\end{align*}
\]

The path \( P(\theta) \), or \( P_i(\theta) \) for \( i = 1, 2 \) when \( P(\theta) = P_1(\theta) + P_2(\theta) \), is the path of steepest descent with respect to the relevant saddle point or points. Both \( p(\omega, \theta) \) and \( P(\theta) \) are continuous functions of \( \theta \) for all \( \theta \in \mathcal{R} \). The uniform asymptotic behavior of the integral \( I(z, \theta) \) for all \( \theta \in \mathcal{R} \) as \( z \to \infty \) is then given by

\[
I(z, \theta) = \exp[q(\omega_0)z] \cdot 
\begin{align*}
&\times \left[ \frac{1}{2} \left( q(\omega_0)h_1(\theta) + q(\omega_0)h_2(\theta) + \mathcal{O} \left( \frac{1}{z} \right) \right) \right] \\
&+ \frac{2\pi i}{z^{3/2}} \exp[q(\omega_0)z] \frac{1}{2 \alpha_1^{1/2}(\theta)} \left[ q(\omega_0)h_1(\theta) - q(\omega_0)h_2(\theta) + \mathcal{O} \left( \frac{1}{z} \right) \right],
\end{align*}
\tag{B4}
\]

where the function \( \mathcal{E}(\gamma) \) is given by

\[
\mathcal{E}(\gamma) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left[ \gamma v - \frac{1}{3} \gamma^3 \right] dv.
\tag{B5}
\]

End of theorem.

The path of integration \( \mathcal{L} \) appearing in Eq. (B5) is the path mapped into the complex \( \nu \) plane as \( \omega \) traverses the contour \( P(\theta) \) by the root of the equation

\[
\frac{1}{\nu} \nu^3 - \alpha_1(\theta)\nu - \alpha_0(\theta) + p(\omega, \theta) = 0
\tag{B6}
\]

that satisfies

\[
\left( \frac{d\nu}{d\omega} \right)_{\omega = \omega_0} = \frac{1}{h_1(\theta)}
\tag{B7}
\]

when \( \theta = \theta_c \). The coefficients appearing in Eqs. (B4), (B6), and (B7) are given by

\[
\begin{align*}
\alpha_0(\theta) &= (1/2)[p(\omega_1, \theta) + p(\omega_2, \theta)],
\alpha_1^{1/2}(\theta) &= \frac{1}{2} \left[ \left( p(\omega_1, \theta) - p(\omega_2, \theta) \right)^{1/2} \right],
\alpha_2(\theta) &= \frac{1}{2} \left[ \left( p(\omega_1, \theta) \right)^{1/2} \right],
\end{align*}
\tag{B8}
\]

for \( \theta \neq \theta_c \). At the critical value \( \theta_c \) of \( \theta \) at which the two saddle points coalesce, these coefficients take on the limiting values

\[
\begin{align*}
&\text{for } \nu = \nu_0: \quad &\frac{2}{h_0(\nu)} \left( q(\omega_0)h_1(\theta) - q(\omega_0)h_2(\theta) \right) = h_3(\theta)q^{(1)}(\omega_0),
&\lim_{\theta \to \theta_c} \left[ \frac{\alpha_1^{1/2}(\theta)}{\omega_0(\theta) - \omega_0(\theta)} \right] &= \frac{1}{2h_1(\theta)}.
\tag{B11}
\end{align*}
\]

and \( \alpha_0(\theta) \) is given by Eq. (B8). The cube root appearing in Eq. (B11) is made single valued by the requirement that

\[
\arg[h_0(\nu)] = \hat{\alpha}_c
\tag{B14}
\]

and \( \hat{\alpha}_c \) is the angle of slope of the path \( P(\theta) \) as it leaves the second-order saddle point at \( \omega = \omega_0 \) for \( \theta = \theta_c \). The cube root appearing in Eq. (B9) is made single valued by the requirement that

\[
\lim_{\theta \to \theta_c} \left[ \frac{\alpha_1^{1/2}(\theta)}{\omega_0(\theta) - \omega_0(\theta)} \right] = \frac{1}{2h_1(\theta)}. \tag{B15}
\]

The square roots appearing in Eqs. (B10a) and (B10b) are made single valued by the requirement that Eq. (B11) be satisfied with the argument of \( h_1(\theta) \) specified by Eq. (B14).

The requirement on the argument of \( h_1(\theta) \) given in Eq. (B14) is the same as that obtained when the more general condition used in Olver's method is applied to the special case in which \( z \) is real and positive and \( P(\theta) = \) the path of steepest descent for \( \theta = \theta_0 \) (at which \( \mu = 3 \)). The above theorem can also be extended to complex \( z \) and arbitrary Olver-type paths. Nevertheless, the theorem is sufficient as stated for the problems treated in this paper.

Consider now the argument of the coefficient \( \alpha_1^{1/2}(\theta) \) given in Eq. (B9). If the saddle point \( \omega_2 \) encircles \( \omega_0 \) once as \( \theta \) varies over \( \mathcal{R} \), then the argument of \( \alpha_1^{1/2}(\theta) \) varies over a range of \( 6\pi \), and the argument of \( \alpha_1^{1/2}(\theta) \) varies over \( 2\pi \).
Hence the cube root appearing in Eq. (B9) is not confined to a single branch of the cube root of $\alpha^{1/2}(\theta)$, as it would be if a branch cut were used to restrict the argument of $\alpha^{1/2}(\theta)$ to a range of less than $2\pi$. To determine the appropriate argument of $\alpha^{1/2}(\theta)$ implied by Eq. (B15), it is useful to apply the following geometrical concepts. Let $\delta_{12}$ be the angle of slope of the vector from $\omega_2$ to $\omega_1$ in the complex $\omega$ plane. Then, according to Eqs. (B14) and (B15),

$$\lim_{\theta \to \theta_s} \arg[\alpha^{1/2}(\theta)] = \delta_{12} - \delta_s + 2\pi n,$$

(B16)

where $n$ is an arbitrary integer. Hence, as $\omega_1$ approaches $\omega_2$ along a straight line, the argument of $\alpha^{1/2}(\theta)$ approaches $2\pi n$ plus the angle that the line makes with the vector tangent to the path of steepest descent as it leaves the saddle point $\omega_2$ when $\theta = \theta_s$. If desired, the integer $n$ can be chosen so that the argument of $\alpha^{1/2}(\theta)$ lies within the principal range $(-\pi, \pi)$ for all $\theta \in \mathcal{R}$. However, it is found to be most convenient to set $n = 0$.

Consider now the argument of the coefficient $h_i(\theta)$, $i = 1, 2, 3$.

When $n$ is the integer appearing in Eq. (B16). It is most convenient to take $n = 0$, since then

$$\lim_{\theta \to \theta_s} \arg[h_i^2(\theta)] = 2\delta_s + 2\pi n,$$

(B19)

where $n$ is the integer appearing in Eq. (B16). It is most convenient to take $n = 0$, since then

$$\arg[h_i(\theta)] = (1/2)\arg[h_i^2(\theta)], \quad i = 1, 2, 3.$$

(B20)

Consideration is given finally to the determination of the contour $\mathcal{L}$ into which the path $P(\theta)$ is mapped under the cubic transformation [Eq. (B6)]. This equation maps the path $P(\theta)$ into three contours in the complex $v$ plane. It follows from Eqs. (B14) and (B17) that the path of integration $\mathcal{L}$ appearing in Eq. (B5) is the one contour of the three that satisfies

$$\left(\frac{dv}{d\omega}\right)_{v = v_0} = -\delta_s,$$

(B21)

when $\theta = \theta_s$.

Since the integrand in Eq. (B5) is an entire function of complex $\alpha$, the only features of the path $\mathcal{L}$ that affect the value of $\mathcal{L}(\theta)$ are the end points of the path. For the integral treated in this paper, the end points are at infinity since $|P(\omega, \theta)| \to \infty$ as $|\omega| \to \infty$ in each direction along $P(\theta)$. Consequently, it follows from Eq. (B6) that the arguments of the end points of $\mathcal{L}$ can lie only in the following regions:

Region 1: $-\pi/6 < \arg(v) < \pi/6$,

(B22a)

Region 2: $\pi/2 < \arg(v) < 5\pi/6$,

(B22b)

Region 3: $-5\pi/6 < \arg(v) < -\pi/2$,

(B22c)

as illustrated in Fig. 11.
APPENDIX C: UNIFORM ASYMPTOTIC EXPANSION FOR A FIRST-ORDER SADDLE POINT AND A SIMPLE POLE SINGULARITY OF THE INTEGRAND

Consider the contour integral

\[ I(z, \theta) = \int_{P'} q(\omega) \exp[zp(\omega, \theta)] d\omega, \]  

where the path of integration \( P' \) extends from \( |\omega| = \infty \) through the finite complex \( \omega \) plane and back to \( |\omega| = \infty \) without forming a closed contour. Let the complex phase function \( p(\omega, \theta) \) be a continuous function of a real parameter \( \theta \) that varies over a domain \( \mathcal{R} \), and let the complex amplitude function \( q(\omega) \) and the contour \( P' \) be independent of \( \theta \). Let \( P(\theta) \) be an Olver-type path with respect to a saddle point \( \omega_{sp}(\theta) \) of \( p(\omega, \theta) \) and let the original contour \( P' \) be deformable to \( P(\theta) \), which moves continuously in the complex \( \omega \) plane as \( \theta \) varies continuously over \( \mathcal{R} \). If no pole singularities of the function \( q(\omega) \) are crossed when \( P' \) is deformed to \( P(\theta) \), then the integral \( I(z, \theta) \) is equal to the integral \( I_{sp}(z, \theta) \), defined by

\[ I_{sp}(z, \theta) = \int_{P(\theta)} q(\omega) \exp[zp(\omega, \theta)] d\omega. \]  

Moreover, if any pole singularities of \( q(\omega) \) are crossed when \( P' \) is deformed to \( P(\theta) \), then the integral \( I(z, \theta) \) is equal to the integral \( I_{sp}(z, \theta) \), defined by

\[ I(z, \theta) = I_{sp}(z, \theta), \quad \theta < \theta_{s}, \]  

\[ I(z, \theta) = I_{sp}(z, \theta) - \pi i \gamma \exp[zp(\omega_{s}, \theta)], \quad \theta = \theta_{s}, \]  

\[ I(z, \theta) = I_{sp}(z, \theta) - \pi i \gamma \exp[zp(\omega_{s}, \theta)], \quad \theta > \theta_{s}, \]  

where

\[ \gamma = \lim_{\omega \to \omega_{s}} \frac{1}{(\omega - \omega_{s})q(\omega)} \]  

is the residue of the simple pole singularity at \( \omega = \omega_{s} \).

To avoid the discontinuous behavior displayed in Eqs. (C3) when the saddle point is near the pole, it is necessary to apply a technique known as subtraction of the pole. This method yields an asymptotic expansion of the integral \( I_{sp}(z, \theta) \) as \( |z| \to \infty \) that is uniform for all \( \theta \in \mathcal{R} \). The resultant asymptotic approximation of the integral \( I(z, \theta) \) is a continuous function of \( \theta \in \mathcal{R} \) for fixed finite \( z \).

The asymptotic behavior of the saddle point integral \( I_{sp}(z, \theta) \) can be stated most simply if \( z \) is taken to be real and positive and if \( P(\theta) \) is taken to be the path of steepest descent through the saddle point. The result, provided in the following theorem by Bleistein,

\[ \textbf{Theorem (Bleistein): In the integral } I_{sp}(z, \theta) \text{ given in Eq. (C2), let the contour of integration } P(\theta) \text{ be the path of steepest descent through a first-order saddle point } \omega_{sp}(\theta) \text{ that is isolated from any other saddle points of } p(\omega, \theta) \text{ and let } z \text{ be real and positive. In addition, let all the conditions required in order for } P(\theta) \text{ to be an Olver-type path with respect to } \omega_{sp}(\theta) \text{ be satisfied for all } \theta \in \mathcal{R} \text{ with the exception that the function } q(\omega) \text{ exhibits a single first-order pole singularity at } \omega = \omega_{s} \text{ in the domain } D \text{ wherein } p(\omega, \theta) \text{ is analytic. The complex phase function } p(\omega, \theta) \text{ is assumed to be a continuous} \]
function of \( \theta \) for all \( \theta \in \mathcal{R} \), whereas \( q(\omega) \) and \( z \) are independent of \( \theta \). Then the saddle point integral \( \text{I}_{sp}(z, \theta) \) satisfies

\[
\text{I}_{sp}(z, \theta) = q(\omega_p) \left[ - \frac{2\pi}{z p^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)]
+ \frac{1}{\Delta(\theta)} \left( \frac{\pi}{z} \right)^{1/2} \exp[zp(\omega_p, \theta)]
+ \frac{\gamma}{\Delta(\theta)} \left[ \pi \right]^{1/2} \exp[zp(\omega_p, \theta)]
+ R_1 \exp[zp(\omega_p, \theta)],
\]

where \( R_1 = a(z^{-3/2}) \) (C5a)

\[
I_{sp}(z, \theta) = q(\omega_p) \left[ - \frac{2\pi}{z p^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)]
+ \frac{1}{\Delta(\theta)} \left( \frac{\pi}{z} \right)^{1/2} \exp[zp(\omega_p, \theta)]
- \frac{\gamma}{\Delta(\theta)} \left[ \pi \right]^{1/2} \exp[zp(\omega_p, \theta)]
+ R_1 \exp[zp(\omega_p, \theta)], \quad \Delta(\theta) = 0,
\]

where \( R_1 = 0(z^{-3/2}) \) (C5b)

\[
I_{sp}(z, \theta) = q(\omega_p) \left[ - \frac{2\pi}{z p^{(2)}(\omega_p, \theta)} \right]^{1/2} \exp[zp(\omega_p, \theta)]
\]

\[
\left[ q(\omega_p) - \frac{\gamma}{\omega_p - \omega_c} - \frac{\gamma}{6p^{(2)}(\omega_p, \theta)} \right]
+ R_1 \exp[zp(\omega_p, \theta)], \quad \Delta(\theta) = 0, \quad \Delta(\theta) \neq 0,
\]

(C5c)

where \( z \to \infty \) uniformly with respect to \( \theta \) for all \( \theta \in \mathcal{R} \). Here \( \gamma \) is defined as in Eq. (C4), and

\[
\Delta(\theta) = \left| p[\omega_p(\theta), \theta] - p(\omega_c, \theta) \right|^{1/2}.
\]

The argument of the quantity \( -zp^{(3)}[\omega_p(\theta), \theta]^{1/2} \) is defined to be equal to \( \arg \left( d\omega \right) \omega_p \), where \( d\omega \) is an element along the path of steepest descent through the saddle point \( \omega_p(\theta) \) and the argument of \( \Delta(\theta) \) is defined so that

\[
\lim_{\omega_p(\theta) \to \omega_c} \frac{\Delta(\theta)}{\omega_p(\theta)} = \left[ \omega_c - \omega_p(\theta) \right] \left[ - \frac{1}{2} p^{(2)}[\omega_p(\theta), \theta] \right]^{1/2}.
\]

Finally, the function \( \text{erfc}(\zeta) \) is the complementary error function, defined by

\[
\text{erfc}(\zeta) = \frac{2}{\pi^{1/2}} \int_{\zeta}^\infty \exp(\xi^2)d\xi.
\]

End of theorem.

The asymptotic behavior of the saddle-point integral \( \text{I}_{sp}(z, \theta) \) is given in the above theorem by Eq. (C5a) with the upper signs when the contour \( P(\theta) \) lies on one side of the pole and with the lower signs when \( P(\theta) \) lies on the other side of the pole with respect to the original path \( P \). When the pole lies on the contour \( P(\theta) \), \( \text{I}_{sp}(z, \theta) \) satisfies Eq. (C5b), and when the saddle point coalesces with the pole, \( \text{I}_{sp}(z, \theta) \) satisfies Eq. (C5c). Since Eq. (C6) for the error term is satisfied uniformly with respect to \( \theta \) for all \( \theta \in \mathcal{R} \), then the apparent discontinuities in the asymptotic behavior of \( I(z, \theta) \) exhibited in Eqs. (C5) are real. In particular, when the path \( P(\theta) \) passes from one side of the pole to the other, the discontinuous jump in \( \text{I}_{sp}(z, \theta) \) that is due to the change in sign of \( \text{Im}(\Delta(\theta)) \) in Eq. (C5a) is \( 2\pi i \gamma \exp[zp(\omega_c, \theta)] \). This discontinuity in \( \text{I}_{sp}(z, \theta) \) exactly cancels the discontinuity in \( I(z, \theta) \) that is introduced by the contribution of the simple pole singularity when Cauchy’s residue theorem is applied to deform the original contour \( P \) to the path of steepest descent \( P(0) \), as exhibited in Eq. (C3). As a result, the asymptotic behavior of \( I(z, \theta) \) is a continuous function of \( \theta \) for all \( \theta \in \mathcal{R} \) for fixed finite values of \( z \).

If \( P(\theta) \) is an Olver-type path other than the path of steepest descent, as used in Ref. 5, then the above theorem remains valid if \( P(\theta) \) can be deformed to the path of steepest descent without crossing the pole singularity. If the pole is crossed when \( P(\theta) \) is deformed to the path of steepest descent, then Eqs. (C5) are changed by the addition or subtraction of a term \( 2\pi i \gamma \exp[zp(\omega_c, \theta)] \) when the change in the expression for \( \text{I}_{sp}(z, \theta) \) is equal to but with a sign opposite that of the change introduced between \( I(z, \theta) \) and \( I_{sp}(z, \theta) \) when Cauchy’s residue theorem is applied to change the contour of integration from the path of steepest descent to the new Olver-type path \( P(\theta) \), the resulting asymptotic expression for \( I(z, \theta) \) remains unchanged. Hence the asymptotic approximation obtained for \( I(z, \theta) \) is independent of the particular Olver-type path chosen. Nevertheless, in order to apply the above theorem to obtain a uniform asymptotic approximation of \( I(z, \theta) \), it is still necessary to determine the path of steepest descent relative to the position of the pole in order to determine whether the contribution of the pole should be added to the right-hand sides of Eqs. (C5).

If the pole encircles the saddle point once as \( \theta \) varies over \( \mathcal{R} \), then the argument of \( \Delta(\theta) \) varies over a range of \( 4\pi \) and the argument of \( \Delta(\theta) \) varies over a range of \( 2\pi \). Hence \( \Delta(\theta) \) is not confined to a single branch of the square root of \( \Delta(\theta) \), as would be obtained by using a branch cut to restrict the argument of \( \Delta(\theta) \) to a range of less than \( 2\pi \). To determine the argument of \( \Delta(\theta) \) implied by Eq. (C8), it is useful to apply the following geometrical concepts. Let \( \alpha_c \) be the angle of slope of the vector from \( \omega_p(\theta) \) to \( \omega_c \) in the complex \( \omega \)-plane. Then Eq. (C8) gives

\[
\lim_{\omega_p(\theta) \to \omega_c} \arg \left[ \frac{\Delta(\theta)}{\omega_p(\theta)} \right] = \alpha_c + \arg \left[ \left( p^{(2)}[\omega_p(\theta), \theta] \right)^{1/2} \right] + 2\pi n,
\]

where the limit is taken along the straight line with slope \( \alpha_c \) and where \( n \) is an arbitrary integer. By the definition given in the above theorem,

\[
\arg \left[ -p^{(2)}[\omega_p(\theta), \theta]^{1/2} \right] = -\alpha SD,
\]

where \( \alpha SD \) is the angle of slope of a vector tangent to the path of steepest descent at the saddle point. Substitution of Eq. (C11) into Eq. (C10) then gives

\[
\lim_{\omega_p(\theta) \to \omega_c} \arg \left[ \frac{\Delta(\theta)}{\omega_p(\theta)} \right] = \alpha_c - \alpha SD + 2\pi n.
\]

Hence, as the pole approaches the saddle point along a straight line, the argument of \( \Delta(\theta) \) approaches \( 2\pi n \) plus the angle that that line makes with the vector tangent to the
path of steepest descent at the saddle point \( \omega_2(\theta) \). The integer \( n \) can then be chosen so that the argument of \( \Delta(\theta) \) lies within the principal range \((-\pi, \pi] \) for all \( \theta \in \mathcal{R} \). The limit of \( \arg(\Delta(\theta)) \) is a small negative angle for the situation depicted in Fig. 12(a) and a small positive angle for those shown in Figs. 12(b) and 12(c).

APPENDIX D: UNIFORM ASYMPTOTIC EXPANSION FOR TWO ISOLATED SADDLE POINTS

Consider a contour integral of the form

\[
I(z, \theta) = \int_{P} q(\omega) \exp[zp(\omega, \theta)] d\omega \tag{D1}
\]

taken over a path \( P' \) that extends from \( |\omega| = \infty \) through the finite complex \( \omega \) plane and back to \( |\omega| = \infty \) without forming a closed contour. Let the complex phase function \( p(\omega, \theta) \) be a continuous function of the real parameter \( \theta \) that varies over a domain \( \mathcal{R} \). Furthermore, let \( \omega_1(\theta) \) and \( \omega_2(\theta) \) denote two isolated first-order saddle points of \( p(\omega, \theta) \) such that

\[
\text{Re}[p(\omega_1, \theta)] > \text{Re}[p(\omega_2, \theta)], \quad \theta < \theta_0, \tag{D2a}
\]

\[
\text{Re}[p(\omega_1, \theta)] = \text{Re}[p(\omega_2, \theta)], \quad \theta = \theta_0, \tag{D2b}
\]

\[
\text{Re}[p(\omega_1, \theta)] < \text{Re}[p(\omega_2, \theta)], \quad \theta > \theta_0. \tag{D2c}
\]

The two saddle points are isolated in that the distance between them is bounded away from zero for all \( \theta \in \mathcal{R} \). In this situation \( \omega_1 \) is called the dominant saddle point for \( \theta < \theta_0 \) and \( \omega_2 \) is said to be dominant for \( \theta > \theta_0 \).

Let the original contour of integration \( P' \) be deformable to a path \( P(\theta) \) that, for all \( \theta \in \mathcal{R} \), passes through both of the saddle points \( \omega_1(\theta) \) and \( \omega_2(\theta) \) and has the following properties. For all \( \theta \in \mathcal{R} \) the contour \( P(\theta) \) moves continuously in the complex \( \omega \) plane as \( \theta \) varies over \( \mathcal{R} \) continuously. Moreover, \( P(\theta) \) can be divided into two parts \( P_1(\theta) \) and \( P_2(\theta) \) such that \( P(\theta) = P_1(\theta) + P_2(\theta) \), where (for \( i = 1, 2 \) \( P_i(\theta) \) passes through the saddle point \( \omega_i(\theta) \) and is an Olver-type path with respect to \( \omega_i(\theta) \). The integral \( I(z, \theta) \) over the contour \( P \) can then be expressed as

\[
I(z, \theta) = I_1(z, \theta) + I_2(z, \theta), \tag{D3}
\]

where

\[
I_i(z, \theta) = \int_{P_i} q(\omega) \exp[zp(\omega, \theta)] d\omega, \quad i = 1, 2. \tag{D4}
\]

It follows from the constraints imposed on the contours \( P_1(\theta) \) and \( P_2(\theta) \) that \( P(\theta) \) is an Olver-type path for the integral \( I(z, \theta) \) with respect to \( \omega_1(\theta) \) when \( \theta < \theta_0 \) and with respect to \( \omega_2(\theta) \) when \( \theta > \theta_0 \). The asymptotic expansion of \( I(z, \theta) \) as \( |z| \to \infty \) is then given by

\[
I(z, \theta) \sim 2 \exp[zp(\omega_1, \theta)] \sum_{n=0}^{\infty} \Gamma(s + \frac{1}{2}) \frac{a_{2n}(1)}{z^{2n+1/2}}, \tag{D5}
\]

where \( \omega_1 = \omega_1(\theta) \) for \( \theta < \theta_0 \) and \( \omega_2 = \omega_2(\theta) \) for \( \theta > \theta_0 \) and the coefficients \( a_{2n}(1) \) are calculated with respect to the saddle point \( \omega_1 \). The discontinuous nature at \( \theta = \theta_0 \) of this asymptotic approximation of \( I(z, \theta) \) as a function of \( \theta \) for fixed \( z \) is obvious. Furthermore, at \( \theta = \theta_0 \) Olver's method cannot be applied to obtain an asymptotic expansion of the integral \( I(z, \theta_0) \).

The discontinuity can be avoided and an expansion at \( \theta = \theta_0 \) can be obtained by applying Olver's method to \( I_i(z, \theta) \) for \( i = 1, 2 \) instead of just applying it to \( I(z, \theta) \). The asymptotic expansion of \( I_i(z, \theta) \) as \( |z| \to \infty \) is given uniformly with respect to \( \theta \) by the right-hand side of relation (D5) for all \( \theta \in \mathcal{R} \). Application of Eq. (D6) consequently yields

\[
I(z, \theta) = 2 \exp[zp(\omega_1, \theta)] \sum_{n=0}^{N-1} \Gamma(s + \frac{1}{2}) \frac{a_{2n}(1)}{z^{2n+1/2}} + O[z^{-(N+1)/2}],
\]

\[
+ 2 \exp[zp(\omega_2, \theta)] \sum_{n=0}^{M-1} \Gamma(s + \frac{1}{2}) \frac{a_{2n}(2)}{z^{2n+1/2}} + O[z^{-(M+1)/2}],
\]

as \( |z| \to \infty \) uniformly with respect to \( \theta \) for all \( \theta \in \mathcal{R} \), where \( N \) and \( M \) are arbitrary positive integers.

For sufficiently large values of \( |z| \) and for fixed \( \theta \neq \theta_0 \), the second term in Eq. (D6) is negligible in comparison with the first term when \( \theta < \theta_0 \), and the first term is negligible in comparison with the second term when \( \theta > \theta_0 \). As a result, Eq. (D6) is equivalent to relation (D5) under these conditions. As \( |\theta - \theta_0| \) tends to zero, however, \( |z| \) must be increased without bound in order for relation (D5) to give a good asymptotic approximation of \( I(z, \theta) \) with a finite, fixed number of terms \( N \). Equation (D6) does not have this difficulty. For sufficiently large fixed values of \( |z| \), Eq. (D6) can be used with fixed values of \( M \) and \( N \) to obtain an asymptotic approximation of \( I(z, \theta) \) that is uniformly valid for all \( \theta \in \mathcal{R} \). The result is clearly a continuous function of \( \theta \).

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