ASYMPTOTIC THEORY OF DISPERSIVE
ACOUSTIC PULSE PROPAGATION

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ABSTRACT

The uniform asymptotic description of the propagation of an ultrashort, plane wave acoustic pulse in a simple dispersive medium with relaxation is considered and compared with its electromagnetic counterpart. The analysis is based on the Fourier integral representation of the solution to the dispersive wave equation

\[
\frac{\partial^2 A}{\partial z^2} - \frac{1}{V^2} \frac{\partial^2 A}{\partial t^2} = -\tau \frac{\partial^2 A}{\partial z^2 \partial t}
\]

for plane wave pulse propagation in the positive z-direction, where V is the constant wave velocity parameter of the medium with relaxation time \( \tau \). The complex wavenumber associated with monochromatic plane wave propagation in the medium is found to satisfy the dispersion relation

\[
\tilde{k}(\omega) = \frac{\omega/V}{(1-i\omega \tau)^{1/2}} = \frac{\omega}{V} n(\omega),
\]

which is similar (but not identical) to that obtained for the classical Debye model for a simple polarizable dielectric in electromagnetic wave theory, where \( n(\omega) \) is identified as the complex index of refraction of the dispersive medium. The asymptotic behavior of \( A(z, t) \) for large values of the propagation distance \( z \) is then described by the relevant saddle points of the complex phase function \( \Phi(\omega, \theta) \), defined by

\[
\Phi(\omega, \theta) = \frac{1}{z} V (\tilde{k}(\omega)z - \omega t) = i\omega (n(\omega) - \theta),
\]

where $\theta = Vt/z$ is a dimensionless space-time parameter, as well as by the behavior of $f(\omega)$ in the region between the original contour of integration of the Fourier integral representation of $A(z, t)$ and these saddle points. As in the electromagnetic case, the asymptotic behavior of the impulse response is found to be described by a single saddle point that begins at $\omega = +i\infty$ and moves down the imaginary axis and approaches the branch point $\omega_p = -i/\tau$ as $\theta$ goes to infinity. In the electromagnetic case the field vanishes for $\theta < 1$ and the saddle point is at $\omega = +i\infty$ at $\theta = 1$. In the acoustic case the field vanishes for $\theta < 0$ and the saddle point is at $\omega = +i\infty$ at $\theta = 0$. This saddle point contribution to the asymptotic behavior of the propagated field produces the so-called Brillouin precursor which has zero attenuation at the space-time point $\theta_0 = n(0)$. This zero attenuation point dominates the propagated field evolution and propagates with the velocity $c/n(0)$ in the electromagnetic case, while in the acoustic case it propagates with the velocity $V$. 