Intracavity adaptive optic compensation of phase aberrations. I: Analysis

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The analytical treatment of linear intracavity phase aberrations and their compensation by an appropriately configured intracavity adaptive optic element is presented in this paper. The approximate geometrical treatment of the effects of intracavity phase aberrations is first treated, followed by a complete exact diffraction integral formulation of the adaptive compensation of linear phase-tilt and phase-curvature aberrations. This new analysis yields the exact intracavity phase compensation for phase tilt, curvature-of-field, and astigmatism-phase aberrations in positive branch, confocal unstable resonator cavities.

The mode structure properties of aligned linear unstable resonator cavities have been exhaustively studied by both numerical and, more recently, analytical techniques. That analysis has yielded both the transverse mode structure of the cavity and the associated complex eigenvalue behavior as a function of both the geometric magnification M and the equivalent Fresnel number Neq of the resonator. An equally important property of a given resonator cavity is the influence of any intracavity gain-phase perturbations on the characteristic mode structure of the cavity. Such intracavity perturbations arise owing to any index of refraction or gain inhomogeneities present in the active gain medium of the laser and any misalignment or surface figure errors in the optical components that make up the resonator cavity. The types of intracavity perturbations present in laser resonators may be separated into two distinct categories: linear (field-independent) phase aberrations and nonlinear (field-dependent) gain-phase perturbations. The field-dependent perturbations, which include such important nonlinear phenomena as gain saturation and resonant dispersion effects, are best treated on an individual basis. The present analysis is restricted to the relatively simpler problem of linear field-independent phase perturbations. An understanding of the effects of such linear phase aberrations on the mode structure properties of unstable resonator cavities and their compensation by adaptive optic techniques is a prerequisite for understanding the effects of nonlinear phase and gain perturbations on the active cavity mode structure and their compensation by novel optical techniques.

The starting point for the analysis given here is an extension of Anan’ev’s geometrical analysis of intracavity phase perturbations. That simple geometric approach yields important results relating the resultant outcoupled phase aberration that is due to a given intracavity phase perturbation of arbitrary order in terms of a simple geometric phase-weighting coefficient. As is shown later, however, these results are valid only for small values of the applied aberration strength. In order to extend the range of applicability of the theory developed here to arbitrarily large aberration strengths, an exact scalar-wave treatment of the required compensation for a given phase tilt or phase curvature aberration is considered next. In future papers, this new analysis will be applied in a numerical study of the resultant deleterious effects of intracavity phase perturbations on the mode structure and beam quality properties of both passive and active linear unstable resonator cavities and their compensation by an ideally configured intracavity adaptive optic element.

1. GEOMETRICAL ANALYSIS OF INTRACAVITY PHASE PERTURBATIONS

The geometrical analysis of linear intracavity phase perturbations, given by Anan’ev, is now extended to more general perturbation situations. That approximate first-order analysis obtains the resultant effect of a given cavity perturbation on the converged phase structure of the resonator cavity. From that analysis is obtained important information regarding the sensitivity of the optical cavity to linear phase aberrations of arbitrary order. Since one is concerned here only with the resultant geometrical effects of an intracavity phase perturbation on the outcoupled mode structure and not with the particular form of the cavity mode structure (which depends on the transverse geometry of the cavity), the results obtained in a simpler one-dimensional analysis are equally applicable to other resonator geometries.

Consider then the x-meridional plane of a linear positive branch, confocal unstable resonator cavity, as shown in Fig. 1. In that figure is depicted the optical path traversed by a typical ray associated with the dominant plane-wave geometric eigenmode of the unperturbed ideal cavity. The effect of any intracavity phase aberration on the phase front of the dominant geometric eigenmode of the cavity is both to alter the paths taken by the geometric rays propagating out from the optic axis of the cavity and to change the optical path length of a given ray from its unperturbed value. The basic simplifying assumption of Anan’ev’s geometrical analysis is that, for the perturbed cavity, the ray paths coincide with the ray paths followed in the ideal unperturbed resonator cavity. The primary effect of the cavity perturbation is to alter the path length along the ray, the deviation of the ray itself from the ideal unperturbed trajectory being a secondary effect. As is evident, this approximation is accurate only for small per-
Fig. 1. Depiction of the optical path traversed by a typical ray associated with the dominant plane-wave geometrical eigenmode of the unaberrated ideal unstable cavity with geometrical magnification $M$. The transverse distance of the ray from the optic axis for the final iterations in the cavity is indicated at the left of the figure.

Fig. 2. Segmentation of an extended inhomogeneous linear phase medium of refractive index $n(x, z)$ into discrete phase planes located at $z_j$.

discretely located phase planes, as illustrated in Fig. 2, with free-space propagation taking place between each of these planes. Let each of these phase planes be axially separated by the distance $\Delta z$ and let the plane located at $z_j$ take into account the medium inhomogeneities occurring in the phase segment extending from $z_j - \Delta z/2$ to $z_j + \Delta z/2$. The number of phase planes necessary to describe the extended inhomogeneous phase medium accurately is bounded below by the requirement that the phase shift per segment be small in comparison to unity. Let $n(x, z)$ denote the real index of refraction of the linear inhomogeneous phase medium. The accumulated phase-perturbation function at the $j$th phase plane is then given by

$$\phi_j(x) = \int_{z_j - \Delta z/2}^{z_j + \Delta z/2} n(x, z) \, dz - \Delta z$$

The phase perturbation at each phase plane within the optical cavity, including that at each intracavity optic element, may then be individually expanded in a Taylor series in the transverse coordinate about the optic axis of the aberration-free cavity, viz.,

$$\phi_j(x) = \sum_{k=1}^{m} \delta_j x^k,$$

where the subscript $j$ denotes the particular phase-perturbation plane. In order to simplify the ensuing analysis, these phase-perturbation (and, possibly, phase-correction) planes will be evenly spaced throughout the optical cavity. As is illustrated in Fig. 3, the phase planes are consecutively numbered in $j$ from the convex feedback mirror, with $j = 0$ corresponding to the convex mirror plane and $j = N + 1$ corresponding to the concave mirror plane, where there is a total of $N$ such intracavity phase planes (discounting those at the convex and concave mirror planes).

The change in optical path length for a single round-trip iteration through the perturbed optical cavity is given by

$$\Delta \xi_{TOT}(x) = \sum_{j=0}^{N+1} \Delta \xi_j \left( \frac{x}{M^2} \right),$$

where $x$ is the transverse coordinate position on the phase front of the exiting cavity mode. The calculation of $\Delta \xi_{TOT}(x)$ then consists of determining $\Delta \xi_j(x)$ for a single round-trip iteration through the optical cavity and then summing the series in Eq. (1.1). Anan’ev\textsuperscript{11} treated the case of a uniformly distributed phase medium (i.e., medium inhomogeneities that are independent of the distance $z$ along the optic axis of the cavity) occupying the cavity. The more general situation of an extended inhomogeneous phase medium with an arbitrary number of intracavity optical elements is now examined.

Consider a linear confocal positive branch unstable resonator cavity defined by a convex feedback mirror and a concave mirror and made up of additional intracavity optical elements and an extended inhomogeneous linear phase medium. Any surface aberrations appearing on the intracavity optical elements may be represented by appropriate phase-perturbation planes at their respective axial locations in the optical cavity. The extended inhomogeneous phase medium may be modeled in the same manner by segmenting it into several phase perturbation strengths and/or for cavities with a large geometric magnification.

The deformation of the exiting cavity mode phase front resulting from the cumulative effect of the intracavity phase aberrations is then equal to the differences between the optical path lengths along the appropriate ray trajectories. If we denote the change in optical path length of the ray during a single round-trip iteration that is due to any phase inhomogeneities present in the cavity by $\Delta \xi(x)$, the optical path difference accrued on the previous round-trip iteration is seen to be given by $\Delta \xi(x/M)$. The total optical path difference is then given by the summation

$$\Delta \xi_{TOT}(x) = \frac{x}{M^2} \sum_{j=0}^{N+1} \Delta \xi_j \left( \frac{x}{M} \right),$$

On substitution of this result into Eq. (1.1), the following expression for the total optical path difference of the phase of the exiting perturbed cavity mode is obtained:

$$\Delta \xi(x) = z_T \sum_{j=0}^{N+1} \sum_{k=1}^{m} \delta_{jk} x^k + \sum_{k=1}^{m} \delta_{jk} x^k \left[ 1 + (M-1) \frac{j}{N+1} \right] \times \left[ 1 + \left( M - 1 \right) \frac{j}{N+1} \right].$$

Fig. 3. Linear positive branch, confocal unstable resonator cavity with a segmented inhomogeneous phase medium and intracavity optical elements.
The geometric phase-weighting factor $\bar{a}_{jk}$ appearing in this expression is given by

$$\bar{a}_{jk} = \frac{1}{M^k - 1} \left[ M^k + \left( 1 + (M - 1) \frac{j}{N + 1} \right)^k \right].$$

(1.5b)

Taking note of the fact that $j/(N + 1) = \frac{z_j}{z_T}$, where $z_j$ is the axial distance of the $j$th phase plane from the convex feedback mirror plane, we see that the requirement that the phase planes be evenly spaced throughout the optical cavity may be dropped. With this substitution, Eqs. (1.5) may then be written in the more general form,

$$\Delta \zeta_{\text{TOT}}(x) = z_T \sum_{j=0}^{N+1} \sum_{k=1}^{m} \delta_{jk} x^k \times \left[ 1 + \frac{1}{M^k} \left[ 1 + (M - 1) \frac{j}{N + 1} \right]^k \right] = z_T \sum_{j=0}^{N+1} \sum_{k=1}^{m} \delta_{jk} \bar{a}_{jk} x^k.$$

(1.5a)

Note that

$$\bar{a}_{jk}(z_T) = 1 + \bar{a}_{jk}(0);$$

that is, the geometric phase-weighting factor at the concave mirror is exactly one greater than that at the convex feedback mirror plane for each order of aberration.

It is readily evident from Eq. (1.6b) that the closer a given phase plane is located to the concave mirror plane, the more pronounced is its effect on the outcoupled phase behavior of the cavity field. That is, the relative sensitivity of the optical cavity to a given order of perturbation monotonically increases as that perturbation is moved through the cavity from the convex feedback mirror plane ($z_j = z_0 = 0$) to the concave mirror plane ($z_j = z_{N+1} = z_T$), as is illustrated in Fig. 4. Note that this axial position dependence of the perturbation is more pronounced for larger cavity magnifications $M$ and perturbation orders $k$. As a consequence, it is best to keep any intracavity phase aberrations removed from the concave mirror. Furthermore, an adaptive phase-correction plane (such as a deformable mirror) will be most effective when placed as near as possible to the concave mirror plane. Any phase errors in this compensating optic element will, however, be most noticeable in this optimum arrangement.

From the general result [Eq. (1.6)], it is seen that a given phase perturbation at the axial location $z_m$ within the resonator cavity may be replaced by an equivalently weighted phase perturbation at a different axial location $z_n$ within the resonator cavity that has the same effect on the exiting phase of the cavity mode as does that at $z_m$ (within the limitations of this geometrical theory). The desired equivalence relation is

$$\delta_k(z_n) = \frac{\bar{a}_{jk}(z_m)}{\bar{a}_{jk}(z_n)} \delta_k(z_m).$$

(1.8)

From Eq. (1.6b), the weighting factor appearing in this expression is seen to be given by

$$\bar{a}_{jk}(z_n) = \frac{M^k + \left[ 1 + (M - 1) \frac{z_n}{z_T} \right]^k}{M^k + \left[ 1 + (M - 1) \frac{z_m}{z_T} \right]^k},$$

(1.9)

where $0 \leq z_m \leq z_T$ and $0 \leq z_n \leq z_T$. For $z_m < z_n$, this weighting factor is between zero and unity, and for $z_m > z_n$, it is greater than unity.

If a negative sign is incorporated on the right-hand side of Eq. (1.8), viz.,

$$\delta_k(z_n) = -\frac{\bar{a}_{jk}(z_m)}{\bar{a}_{jk}(z_n)} \delta_k(z_m),$$

(1.10)

then the phase structure applied to the intracavity mode at the plane $z_n$ will compensate for the phase perturbation applied at the plane $z_m$. Notice that the various orders of perturbation are not coupled together in this geometrical theory. It is intuitively evident, however, that a finite amount of coupling occurs under propagation between the various orders of aberration imposed on the phase front of the wave field. The geometrical analysis developed here does not describe this coupling because of the initial simplifying assumption that the perturbed ray paths coincided with the ideal unperturbed ray paths of the optical cavity. In Section 2 of this paper, this restriction is removed in a complete scalar-wave diffraction analysis of the adaptive compensation of phase-tilt and phase-curvature aberrations.

The preceding discretized analysis of an inhomogeneous intracavity phase medium may be extended in a straightforward manner to one in which the continuous nature of the medium is preserved. For the simple case in which the re-
fractive index of the phase medium is independent of the location \( z \) along the optic axis and is uniformly distributed throughout the entire optical cavity, the resultant total optical path difference of the phase of the exiting perturbed cavity mode is

\[
\Delta \xi_{\text{hom}}(x) = z_T \sum_{k=1}^{m} n_k \bar{\alpha}_k x^k, \quad (1.11)
\]

which was first obtained by Anan’ev.\(^1\) The geometrical phase-weighting factor appearing in this expression is given by

\[
\bar{\alpha}_k = \frac{1}{M^k - 1} \left[ M^k + \frac{M^{k+1} - 1}{(k+1)(M-1)} \right], \quad k = 1, 2, 3, \ldots, \quad (1.12)
\]

Since the geometrical phase-front distortions that are accumulated in a single pass of a collimated beam through the perturbing medium are given by \( z_T \sum_{k=1}^{n_k} n_k x^k \), the phase-weighting factor \( \bar{\alpha}_k \) is a measure of how many times stronger the influence of a given phase perturbation is in a positive branch, confocal unstable resonator cavity than in a single-pass collimated system. The functional dependence of the geometric phase-weighting factor \( \bar{\alpha}_k \) on the cavity magnification \( M \) is illustrated in Fig. 5 for several values of the perturbation order \( k \). From this figure it is clear that for a positive branch, confocal unstable resonator cavity, the influence of a given phase-medium perturbation decreases monotonically with increasing magnification and that for a given magnification the influence of any phase perturbations decreases monotonically with the order \( k \) of the aberration. For cavity magnifications approaching unity from above, the eigenmode phase structure becomes increasingly sensitive to higher-order phase perturbations present in the cavity medium.

In the most general case of an inhomogeneous phase medium extending along the optic axis from \( z_m \) to \( z_w \) within the optical cavity along with the presence of phase perturbations at any of the optical elements that make up the cavity, the correction coefficients describing the phase profile at a single corrective optical element located at \( z = z_i \) within the cavity are seen to be given by

\[
\delta_k(z_i) = -\frac{1}{\bar{\alpha}_k(z_i)} \times \left[ \sum_{j=0}^{N+1} \delta_k(z_j) \bar{\alpha}_k(z_j) + \frac{1}{z_T} \int_{z_m}^{z_w} n_k(z) \bar{\alpha}_k(z) dz \right]. \quad (1.13)
\]

Again it is seen that, within the geometrical approximation developed here, a given order \( k \) of aberration is not influenced by a different order \( k' \) of aberration.

The above geometrical analysis provides a basic understanding of the influence of intracavity phase perturbations on the phase of the exiting cavity mode structure for a linear positive branch, confocal unstable resonator cavity. The underlying approximation that the perturbed ray paths follow the unperturbed ray trajectories of the ideal unperturbed optical cavity is critical in that analysis in two different ways. First, and most importantly, that simplifying assumption reduced the ensuing analysis to a tractable form from which the influence of a given phase perturbation on the phase of the exiting cavity mode could be directly determined. Second, that assumption naturally limited the accuracy of the resulting analyses to small perturbation strengths. The relative effect of a given phase perturbation on a ray trajectory depends on the order of the aberration. For low-order aberrations, such as tilt \((k = 1)\), curvature of field, and astigmatism \((k = 2)\), the actual perturbed ray trajectory is strongly dependent on the strength of the perturbation. For higher-order perturbations, however, the actual perturbed ray trajectory is deviated from the unperturbed path to a lesser degree than that resulting from the same strength of lower-order aberration. The geometrical analysis is then expected to become more accurate for higher-order perturbation orders.

2. SCALAR-WAVE ANALYSIS OF PHASE-TILT AND PHASE-CURVATURE ABERRATIONS AND THEIR ADAPTIVE COMPENSATION

The application of the preceding geometrical analysis to determine the phase correction necessary to compensate for a given intracavity phase perturbation is severely limited in that approach by the inherent restriction to small perturbation strengths for low-order aberrations. That restriction is removed in this section, wherein an exact diffraction integral formulation of the adaptive compensation of phase-tilt and phase-curvature aberrations is developed. In this formulation the diffraction that is due to the feedback mirror aperture alone is considered, because this is the primary source of edge-diffracted field components into the unstable cavity mode field formation.\(^5,9\) The two aberrations of concern are those to which a positive branch, confocal unstable resonator cavity is most sensitive (see Figs. 4 and 5).

For purposes of definiteness, the scalar-wave diffraction analysis of the intracavity adaptive compensation of a quadratic phase perturbation developed here is restricted to the linear positive branch, confocal unstable resonator cavity configuration illustrated in Fig. 6. The given intracavity...
The correction technique employed here is thus one that is designed to drive the perturbed cavity mode structure back to the ideal unperturbed cavity eigenstructure. It does not attempt in its formulation to optimize the far-field diffraction pattern of the outcoupled cavity mode. However, since it does drive the perturbed mode structure back toward its unperturbed state, it does force the far-field pattern to approach that in the ideal unperturbed case, which may be considered an optimum for a given cavity configuration.

2.1. Pure Phase-Tilt Aberration

Consider first the case of a pure phase-tilt aberration applied in the x-transverse dimension at the intracavity phase-perturbation plane located at a distance \( z_1 \) from the convex feedback mirror plane. The particular form of the phase perturbation is given by

\[
\phi(x) = \delta_1 x, \tag{2.1}
\]

where the dimensionless parameter \( \delta_1 \) represents the total amount of phase-tilt aberration (about the ideal optic axis of the unperturbed cavity) that is applied to the phase front of the scalar-wave field incident upon that plane. For a positive phase-tilt aberration (\( \delta_1 > 0 \)), the phase front of the incident-wave field is advanced for \( x > 0 \) and retarded for \( x < 0 \) with respect to the phase at \( x = 0 \).

Let \( R_c \) and \( R_p \) denote the magnitudes of the unperturbed radii of curvature of the convex mirror, and let \( M_x \) and \( M_y \) denote the geometric magnifications of the unperturbed optical cavity in the \( x \) and \( y \) meridional planes, respectively. The integral equation for the phase-tilt-aberrated cavity with convex mirror feedback aperture \( A \) may be directly obtained in terms of the meridional eikonal of the cavity through the augmented paraxial ray transfer matrix for a single round-trip iteration (see Refs. 12 and 14). The result is

\[
\begin{align*}
\tilde{f} u(x_2, y_2) & = -\frac{i}{2 \lambda z_T} e^{i 2 \pi x z_T} \\
& \times \exp \left\{ -i \frac{2 \delta_1 M_x}{M_x + 1} \frac{z_2 + z_3}{R_x} x_2 \right\} \\
& \times \int_A u(x_1, y_1) \exp \left\{ -i \frac{2 \delta_1 M_x}{M_x + 1} \frac{z_2 + z_3}{R_x} x_1 \right\} \\
& \times \int \exp \left\{ i \frac{\pi}{\lambda z_T} \left[ \frac{(x_2 - M_x x_1)^2}{M_x + 1} + \frac{(y_2 - M_y y_1)^2}{M_y + 1} \right] \right\} dx_1 dy_1. \tag{2.2}
\end{align*}
\]

As is evident, the symmetry of the kernel in the \( x \) dimension has been destroyed by the introduction of the phase-tilt perturbation. The relative effect of the phase-tilt aberration on the cavity mode structure depends on the particular cavity configuration through the magnitude of the quantity

\[
1 - \frac{1}{M_x R_x} (z_2 + z_3) = \frac{1}{M_x} \left( 1 + \frac{z_1 + z_T}{R_x} \right),
\]

which is small for large magnification \( M_x \) and small \( z_1 \). That is, the relative effect of the aberration on the cavity mode decreases for increasing cavity magnification and is minimized with respect to location if the aberration is placed as close as possible to the convex mirror plane, in agreement with the results of the geometrical analysis given in Section 1.

With the application of a compensative phase-tilt surface profile \( \phi(x) = \beta_1 x \) at the intracavity active optic element loc-
cated a distance \( z_2 \) from the phase-perturbation plane, it is found that the required corrective-tilt strength given by Eq. (1.10), viz.,

\[
\beta_1 = -\delta_1 \frac{M_x (2z_1 + z_2 + z_3) + z_2 + z_3}{M_x (2z_1 + 2z_2 + z_3) + z_3},
\]

(2.3)
is exact.

Furthermore, since the phase-tilt coefficients \( \delta_1 \) and \( \beta_1 \) that appear in the augmented ray transfer matrix for the tilt-compensated cavity remain separated (i.e., there are no cross terms in \( \delta_1 \beta_1 \)), it follows that the geometrical analysis of Section 1 for the ideal intracavity correction coefficient is valid for any given distribution of pure phase-tilt aberrations present in the resonator cavity. This may not be true, however, for other more general situations of intracavity aberrations.

In addition, with that ideal correction coefficient applied at the intracavity active optic element, the augmented paraxial ray transfer matrix for the phase-tilt-corrected resonator cavity is found to reduce to the ideal unperturbed ray transfer matrix for that cavity. The resulting integral equation for the phase-tilt-corrected cavity mode is then that of the ideal unperturbed cavity mode within the paraxial approximation.

As a consequence, if the only limiting aperture within the optical cavity is that of the convex feedback mirror, then the resulting corrected cavity mode structure is identical with the ideal unperturbed cavity mode structure. However, if additional diffracting apertures are placed in the cavity between the phase-perturbation plane and the concave mirror plane, a certain degree of vignetting (dependent on the relative size and shape of the aperture and the magnitude of the applied phase-tilt aberration) will occur in the formation of the corrected cavity mode, and the ideal mode properties of the cavity can then only be approached.

2.2. Pure Phase-Curvature Aberration

Consider now the case of a pure phase-curvature aberration applied in the \( x \)-transverse dimension at the intracavity phase-perturbation plane located at an axial distance \( z_1 \) from the convex feedback mirror plane. The particular form of the applied phase perturbation is given by

\[
\psi(x) = \delta_2 x^2,
\]

(2.4)

where the parameter \( \delta_2 \) represents the total amount of phase-curvature aberration (about the ideal optic axis of the unperturbed cavity) that is applied in the \( x \)-transverse dimension to the phase front of the scalar-wave field incident upon that plane. If an equal amount of the same type and sign of perturbation is applied in the \( y \)-transverse dimension, the total aberration applied is a pure curvature of field; if an equal but opposite amount of the same type of perturbation is applied in the \( y \) direction, the total aberration applied is pure astigmatism; if no aberration is applied in the \( y \) direction, the total aberration is a pure cylinder in the \( x \) direction; and if unequal amounts of this type of perturbation are applied in the two orthogonal transverse dimensions, the total aberration is a linear combination of curvature of field, cylinder, and astigmatism. For a positive curvature aberration (\( \delta_2 > 0 \)), the phase front of the incident-wave field is made to converge, whereas for a negative curvature aberration (\( \delta_2 < 0 \)), the phase front is made to diverge. In the paraxial approximation, the radius of curvature applied to the phase of the incident-wave field is given by

\[
R_p = \frac{1}{2\delta_2}.
\]

As in the previous subsection, let \( R_x \) denote the magnitude of the unperturbed radius of curvature of the convex mirror and \( M_x \) denote the geometrical magnification of the unperturbed optical cavity in the \( x \)-meridional plane. The paraxial ray transfer matrix for a single round-trip iteration through the phase-curvature-perturbed resonator cavity is then given by

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{M_x}{M_x + \Omega} & z_T \left( \frac{1}{M_x} + \frac{\Delta}{M_x + \Gamma} \right) \\ \frac{\Gamma}{M_x + \tau} \end{pmatrix},
\]

(2.5)

where

\[
\Omega = 8R_x^2 (2z + 3) \left( 1 + \frac{R_x}{z_x} \right) \left( 1 - \frac{1}{M_x R_x} (z_x + 3) \right)
- 4\delta_2 \left( 1 + \frac{2}{R_x} z_1 \right) \left( z_T - \frac{1}{M_x R_x} (z_T + 1) \right)
+ \frac{2}{R_x} z_1 (z_x + 3) \left( 1 - \frac{1}{M_x R_x} (z_x + 3) \right), \tag{2.6a}
\]

\[
\Delta = 8R_x^2 (2z + 3) \left( 1 - \frac{1}{M_x R_x} (z_x + 3) \right)
- 4\delta_2 z_1 \left( z_T + z_x + 3 - \frac{2}{M_x R_x} z_T (z_x + 3) \right), \tag{2.6b}
\]

\[
\Gamma = 8R_x^2 (2z + 3) \left( 1 + \frac{2}{R_x} z_1 \right) \left( 1 - \frac{1}{M_x R_x} (z_x + 3) \right)
- 4\delta_2 \left( 1 + \frac{2}{R_x} z_1 \right) \left( 1 - \frac{2}{M_x R_x} (z_x + 3) \right)
+ \frac{2}{R_x} (z_x + 3) \left( 1 - \frac{1}{M_x R_x} (z_x + 3) \right), \tag{2.6c}
\]

\[
\tau = 8R_x^2 (2z + 3) \left( 1 - \frac{1}{M_x R_x} (z_x + 3) \right)
- 4\delta_2 \left( z_T - \frac{1}{M_x R_x} (z_T + 1) \right), \tag{2.6d}
\]

The paraxial ray transfer matrix in the \( y \)-meridional plane for a single round-trip iteration through the resonator cavity is unaffected by the phase-curvature aberration applied in the \( x \)-meridional plane. For a similar curvature aberration applied in the \( y \)-meridional plane, the resulting phase-curvature-perturbed paraxial ray transfer matrix in that plane is given by an equation with the same functional form as in Eq. (2.5).

The integral equation that describes the mode structure properties of the phase-curvature-aberrated resonator cavity is
\[
\tilde{y}u(x_2, y_2) = -\frac{i}{2\lambda z_T} e^{ik_2z_T} \int \int u(x_1, y_1) \\
\times \exp \left\{ \frac{k}{2} \left[ (M_x + \Omega_x) x_1^2 + \frac{1}{M_x} x_2^2 - 2x_1x_2 \right] \right. \\
\left. \frac{1}{z_T \left( 1 + \frac{1}{M_x} \right) + \Delta_x} \right\} dx_1 dy_1,
\]

(2.7)

with (in general) differing amounts of phase-curvature aberration applied in the x- and y-meridional planes. In the special case of a convex feedback mirror with rectangular aperture with edges parallel to the x and y coordinate axes, the transverse scalar-wave mode disturbance given by Eq. (2.7) is completely separable in the x- and y-coordinate directions. The integral equation in the x-meridional plane is then given by

\[
\tilde{y}u(x_2) = \left( -\frac{i}{2N_T} \right)^{1/2} e^{ik_2z_T} \int_{-a}^{a} u(x_1) \\
\times \exp \left\{ \pi N_c \left[ 1 + \frac{1}{x_2 M_x} + \frac{1}{1 + \Omega_x/M_x} \right] \right\} dx_1,
\]

with a similar expression holding in the y-meridional plane. Since the quantity \(\Delta x\) appearing in the denominator of the exponential in this expression is negligible in comparison to the quantity \(z_T(1 + 1/M_x)\), it may be ignored. Furthermore, under the change of variables \(x_1 = a\xi\) and \(x_2 = M_x a\xi\), the above expression becomes

\[
\tilde{y}u(M_x a\xi) = \left( -\frac{i}{2N_T} \right)^{1/2} e^{ik_2z_T} \\
\times \exp \left\{ \pi N_c \left[ 1 + \frac{1}{x_2 M_x} \right] \right\} \left[ \frac{1}{1 + \Omega_x/M_x} \right] dx_1,
\]

where \(\Omega_x\) is given by Eq. (2.6a). A similar expression holds in the y-meridional plane. The behavior of \(N_T^p/N_c\) as a function of the number \(m\) of wavelengths of curvature aberration (center to edge) is illustrated in Fig. 8 for a linear confocal positive branch unstable resonator cavity. As can be seen, for positive values of phase-curvature aberration, the cavity Fresnel number is decreased and the outcoupled mode phase front is converging, whereas for negative values the cavity Fresnel number is increased and the outcoupled mode phase front is diverging. Since \(\Omega_x(\pm \beta_2) = \Omega_x(-\beta_2)\), the magnitude of the perturbed cavity Fresnel number is altered by different amounts depending on the sign of the applied phase-curvature aberration. As a consequence, for a pure-astigmatic-perturbed resonator cavity, the aberrated mode structure is primarily astigmatic perturbed, but, in addition, a small but finite amount of curvature-of-field aberration is present. Finally, note that the above expression (Eq. 2.9) for the perturbed-cavity Fresnel number also holds in the case of resonator cavities with circular outcoupling aperture.

Consider now the application of a compensating phase-curvature surface profile \(\phi(x) = \beta_2 x^2\) at the intracavity active optic element located a distance \(z_2\) from the phase-perturbation plane (see Fig. 6), where the proper correction coefficient \(\beta_2\) remains to be determined in terms of the perturbation coefficient \(\delta_2\). The paraxial ray transfer matrix in the x-meridional plane for a single round-trip iteration through the phase-curvature-corrected resonator cavity is then given by

\[
N_T = \frac{a^2}{\lambda z_T},
\]

\[
N_c = \frac{M^2 a^2}{\lambda z_T (M + 1)}.
\]

The perturbed collimated Fresnel number of the phase-curvature-aberrated optical cavity is then seen to be given by

\[
N_T^p = N_c \left[ 1 + \frac{\Omega_x}{M_x} \right],
\]

(2.9)
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \left(\begin{array}{c}
M_x + \Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_3 \\
\Gamma_0 + \Gamma_1 \beta_2 + \Gamma_2 \beta_3
\end{array}\right) \left(\begin{array}{c}
\frac{z_T}{1 + \frac{1}{M_x}} + \Delta_0 + \Delta_1 \beta_2 + \Delta_2 \beta_3 \\
\frac{1}{M_x} + \tau_0 + \tau_1 \beta_2 + \tau_2 \beta_3
\end{array}\right),
\]

(2.10)

where

\[
\Omega_0 = -2\delta_2 \left(\frac{1 + 2}{R_x} z_1\right) \left[z_T + (z_2 + z_3) \left(1 - \frac{2}{M_x} z_T\right)\right] + z_1 \left[\left(1 + \frac{2}{R_x} z_T\right) \left(1 - \frac{2}{M_x} z_T\right) + \frac{2}{R_x} (z_2 + z_3)\right] + 8\delta_2^2 (z_2 + z_3) \left[1 - \frac{1}{M_x} (z_2 + z_3)\right] z_1 \left(1 + \frac{2}{R_x} z_1\right),
\]

(2.11a)

\[
\Omega_1 = -2 \left[\left(1 + \frac{2}{R_x} (z_1 + z_2)\right) \left[z_T + z_3 \left(1 - \frac{2}{M_x} z_T\right)\right] + (z_1 + z_2) \left[\left(1 - \frac{2}{M_x} z_3\right) \left(1 + \frac{2}{R_x} z_T\right) + \frac{2}{R_x} z_3\right]\right] + 4\delta_2 \left(z_2 \left(1 + \frac{2}{R_x} z_1\right) \left[z_T + z_3 \left(1 - \frac{2}{M_x} z_T\right)\right] + z_1 \left[\left(1 + \frac{2}{R_x} z_1\right) \left[z_2 + z_3 + \left(\left(1 - \frac{2}{M_x} z_3\right) \left(1 + \frac{2}{R_x} z_T\right) + \frac{2}{R_x} z_3\right)\right] - 16\delta_2^2 z_2 z_1 \left(1 + \frac{2}{R_x} z_1\right) \times \left[2 z_2 + 2 z_3 \left[1 - \frac{1}{M_x} (z_2 + z_3)\right]\right],
\]

(2.11b)

\[
\Omega_2 = 8\delta_2 \left(1 - \frac{1}{M_x} z_3\right) \left(z_1 + z_2\right) \left[1 + \frac{2}{R_x} (z_1 + z_2)\right] - 2\delta_2^2 \left(z_1 + z_2\right) \left[1 + \frac{2}{R_x} z_1\right] + z_1 \left[1 + \frac{2}{R_x} (z_1 + z_2)\right] + 4\delta_2^2 z_2 \left(1 + \frac{2}{R_x} z_1\right) \left(z_1 + z_2\right),
\]

(2.11c)

\[
\Delta_0 = -2\delta_2 \left(z_1 \left[z_T + (z_2 + z_3) \left(1 - \frac{2}{M_x} z_T\right)\right] + z_1 \left[z_2 + z_3 + z_T \left(1 - \frac{2}{M_x} (z_2 + z_3)\right)\right]\right) + 8\delta_2^2 z_1^2 \left(z_2 + z_3\right) \left[1 - \frac{1}{M_x} (z_2 + z_3)\right],
\]

(2.11d)

\[
\Delta_1 = -2 \left\{\left(z_1 + z_2\right) \left[z_T + z_3 \left(1 - \frac{2}{M_x} z_T\right) + z_3 + z_T \left(1 - \frac{2}{M_x} z_3\right)\right]\right\} + 4\delta_2 \left(z_2 \left[z_T + z_3 \left(1 - \frac{2}{M_x} z_T\right) + z_3 + z_T \left(1 - \frac{2}{M_x} z_3\right)\right]\right\} + (z_1 + z_2) \left[z_2 + z_3 + z_3 \left[1 - \frac{2}{M_x} (z_2 + z_3)\right] + (z_1 + z_2) \left[z_3 + (z_2 + z_3) \left[1 - \frac{2}{M_x} z_3\right]\right] + z_2 \left[z_3 + z_T \left(1 - \frac{2}{M_x} z_3\right)\right] - 16\delta_2^2 z_2 z_1 \left[1 - \frac{1}{M_x} (z_2 + z_3)\right]\right\},
\]

(2.11e)

\[
\Delta_2 = 8z_3 \left(1 - \frac{1}{M_x} z_3\right) \times \left\{\left(z_1 + z_2\right) \left[z_1 + z_2 - 4\delta_2 z_1 z_2 + 4\delta_2^2 z_2^2\right]\right\},
\]

(2.11f)

\[
\Gamma_0 = -2\delta_2 \left[\left(1 + \frac{2}{R_x} z_1\right) \left[1 - \frac{2}{M_x} (z_2 + z_3)\right] + 1 + \frac{2}{R_x} z_T\right] \left[1 - \frac{2}{M_x} (z_2 + z_3) + \frac{2}{R_x} z_2 + z_3\right]\right\} + 8\delta_2^2 \left(z_2 + z_3\right) \left[1 - \frac{1}{M_x} (z_2 + z_3)\right] \left[1 + \frac{2}{R_x} z_1\right],
\]

(2.11g)

\[
\Gamma_1 = -2 \left[\left(1 + \frac{2}{R_x} (z_1 + z_2)\right) \left[1 - \frac{2}{M_x} z_3\right] + \left(1 - \frac{2}{M_x} z_3\right) \left(1 + \frac{2}{R_x} z_T\right) + \frac{2}{R_x} z_3\right] + 4\delta_2 \left(z_2 \left[1 + \frac{2}{R_x} z_1\right] \left[1 - \frac{2}{M_x} z_3\right]\right) + \left[1 + \frac{2}{R_x} (z_1 + z_2)\right] \left\{z_2 + z_3 + z_3 \left[1 - \frac{2}{M_x} (z_2 + z_3)\right]\right\} + \left[1 + \frac{2}{R_x} z_2\right] \left[z_2 + z_3 + z_3 \left[1 - \frac{2}{M_x} z_3\right]\right]\right\},
\]

(2.11h)
\[ \Gamma_2 = 8z_3 \left( 1 - \frac{1}{M_x R_x} z_3 \right) \left( 1 + \frac{2}{R_x} (z_1 + z_2) \right) - 4 \delta_2 z_2 \left( 1 + \frac{1}{R_x} (2z_1 + z_2) \right) + 4 \delta_2^2 z_2 \left( 1 + \frac{2}{R_x} z_1 \right), \]  

\[ \tau_0 = -2 \delta_2 \left\{ z_1 \left( 1 - \frac{2}{M_x R_x} (z_2 + z_3) \right) + z_2 + z_3 + z_T \left( 1 - \frac{2}{M_x R_x} (z_2 + z_3) \right) + 8 \delta_2^2 z_1 (z_2 + z_3) \left[ 1 - \frac{1}{M_x R_x} (z_2 + z_3) \right] \right\}, \]  

\[ \tau_1 = -2 \left\{ \left( z_1 + z_2 \right) \left[ 1 - \frac{2}{M_x R_x} z_3 \right] + z_3 + z_T \left( 1 - \frac{2}{M_x R_x} z_3 \right) + 4 \delta_2 \left( z_1 z_2 \left[ 1 - \frac{2}{M_x R_x} z_3 \right] + (z_1 + z_2) \right) \times \left[ z_2 + z_3 + z_T \left( 1 - \frac{2}{M_x R_x} (z_2 + z_3) \right) \right] + z_1 \left[ z_3 + (z_2 + z_3) \left( 1 - \frac{2}{M_x R_x} z_3 \right) \right] + z_2 \left[ z_3 + z_T \left( 1 - \frac{2}{M_x R_x} z_3 \right) \right] \right\} + 8 \delta_2^2 z_1 z_2 \left\{ z_2 + 2 z_3 \left[ 1 - \frac{1}{M_x R_x} (z_2 + z_3) \right] \right\}, \]  

\[ \tau_2 = 8z_3 \left( 1 - \frac{1}{M_x R_x} z_3 \right) \times \left[ z_1 + z_2 - 2 \delta_2 z_2 (2z_1 + z_2) + 4 \delta_2^2 z_2 z_1 \right]. \]

In the ideal unperturbed cavity configuration a ray incident upon the convex feedback mirror plane with zero optical tangent (which corresponds to the dominant geometric mode of the unperturbed cavity) is, after a single round-trip iteration through the resonator cavity, incident back upon that plane with zero optical tangent at a magnified distance from the optic axis. In the phase-curvature-corrected cavity configuration, it is required that the paraxial ray transfer matrix [Eq. (2.10)] satisfy the equation

\[
\begin{aligned}
\begin{pmatrix}
x_2 \\
x_0
\end{pmatrix} &= \begin{pmatrix}
M_x + \Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_2^2 & x_T \left( 1 + \frac{1}{M_x} \right) + \Delta_0 + \Delta_1 \beta_2 + \Delta_2 \beta_2^2 \\
\Gamma_0 + \Gamma_1 \beta_2 + \Gamma_2 \beta_2^2 & \frac{1}{M_x} + \tau_0 + \tau_1 \beta_2 + \tau_2 \beta_2^2
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_0
\end{pmatrix},
\end{aligned}
\]

which, in turn, implies that the desired correction coefficient be given by

\[ \beta_2 = -\Gamma_1 - (\Gamma_1^2 - 4 \Gamma_0 \Gamma_2)^{1/2} / 2 \Gamma_2, \]  

which is exact within the framework of paraxial optics. The negative sign is used in this expression when the positive branch of the square-root operation is employed. This choice of sign in Eq. (2.12) yields \( \beta_2 = 0 \) when \( \delta_2 = 0 \). A similar quadratic equation also results for the correction coefficient to a phase-curvature aberration applied in the \( y \)-meridional plane.

Now compare this result with that predicted by the geometrical analysis of Section 1. According to Eq. (1.10), the geometrically predicted phase-curvature correction coefficient is given by

\[ \beta_2 (z_1 + z_2) = -\frac{\alpha_2 (z_1)}{\alpha_2 (z_1 + z_2)} \delta_2 (z_1), \]

where the phase-weighting coefficient appearing in this expression is given by

\[ \frac{\alpha_2 (z_1)}{\alpha_2 (z_1 + z_2)} = \frac{M^2 + \left[ 1 + (M - 1) \frac{z_1}{z_T} \right]^2}{M^2 + \left[ 1 + (M - 1) \frac{z_1 + z_2}{z_T} \right]^2}. \]

Notice that this geometrical phase-correction weighting coefficient depends only on the geometrical magnification of the unperturbed cavity configuration and the relative locations of the phase-perturbation and -correction planes within the cavity. It is independent of the phase-curvature perturbation strength \( \delta_2 \). The exact paraxial phase-correction curvature weighting coefficient obtained from Eq. (2.12) by taking the ratio \(-\beta_2 / \delta_2\) depends in a complicated fashion on the unperturbed cavity magnification, the relative locations of the phase-perturbation and -correction planes within the cavity, the unperturbed radii of curvature of the cavity mirrors, and the phase-curvature perturbation strength \( \delta_2 \). It
is intuitively expected that the scalar-wave diffraction result given by Eq. (2.12) will reduce to the approximate geometric result given above in the limit as the phase-curvature perturbation strength $\delta_2$ goes to zero. This is illustrated in Fig. 9, which shows the behavior of the phase-curvature correction weighting factor $-\beta_2/\delta_2$ as a function of the number $m$ of wavelengths of phase-curvature aberration (center to edge) as given by the exact equation [Eq. (2.12)] (as depicted by the solid line in the figure) and the above geometric approximation (as depicted by the dashed line), which is $-\beta_2/\delta_2 = 0.85246$ for the case illustrated of a linear confocal positive branch unstable resonator cavity with $z_1 = z_2 = z_3 = 503.145$ cm, $M = 2$, $R = 3018.868$ cm, and $N_c = 0.8333$. As can be seen, the approximate geometrical result is accurate only for very small values of the perturbation strength. This failure of the geometrical result is indicative of the error inherent in the assumption that the perturbed ray paths coincide with the ideal unperturbed ray trajectories.

The integral equation that describes the mode structure properties of the phase-curvature-corrected resonator cavity is

$$
\gamma u \left[ M_x \mu \left( 1 + \frac{\Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_2^2}{M_x} \right) \right]
= \sqrt{\frac{2}{N_T}} \exp \left( i \pi N_c \frac{1 + \Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_2^2}{M_x} \right) \times \left[ 1 + (r_0 + r_1 \beta_2 + r_2 \beta_2^2) M_x \right]
+ \frac{M_x}{M_x + \Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_2^2} \mu^2 \right)
\times \int_{-1}^{1} u(\xi') \exp \left( i \pi N_c \left( 1 + \frac{\Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_2^2}{M_x} \right) (\xi - \mu)^2 \right) d\xi',
(2.13)
$$

where $\Omega_0, \Omega_1, \Omega_2, r_0, r_1$, and $r_2$ are as defined by Eq. (2.11) and $\beta_2$ is given by Eq. (2.12). The corrected collimated Fresnel number of the phase-curvature-compensated optical cavity is then given by

$$
N_c^r = N_c \left( 1 + \frac{\Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_2^2}{M_x} \right),
(2.14)
$$

where a similar expression holds in the $\xi$-meridional plane. The behavior of $N_c^r/N_c$ as a function of the number $m$ of wavelengths of phase-curvature aberration (center to edge) is depicted by the dashed line in Fig. 8. Notice that, although the corrected cavity is again confocal, the ideal unperturbed cavity Fresnel number is only approached and is not reobtained with this applied correction. For larger cavity magnifications, however, the corrected cavity Fresnel number will approach the ideal unperturbed value more closely but will never attain that value for finite cavity magnifications. To reobtain the ideal unperturbed cavity Fresnel number exactly would entail applying a correction coefficient $\beta_2$ at the intracavity adaptive optic element that satisfied the equation

$$
\Omega_0 + \Omega_1 \beta_2 + \Omega_2 \beta_2^2 = 0.
$$

With this correction applied, however, confocality would not be achieved, and the resulting outcoupled mode phase structure would not be collimated.

The effect of the intracavity phase-curvature perturbation and correction on the dominant geometric mode phase front formed by the cavity in the perturbed and corrected configurations is illustrated in Fig. 10. The application of a negative phase-curvature aberration $\beta_2 < 0$ [see Fig. 10(a)] causes the exiting cavity mode phase front to be divergent, thereby increasing the geometric magnification and Fresnel number of the cavity. The proper correction is to apply an appropriately weighted positive phase curvature at a later plane within the cavity that recollimates the mode in a single iteration, as illustrated. The corrected cavity is again confocal, but with
Furthermore, the mathematically defined ideal correction for element provided that no intracavity vignetting occurs. Aberrations is attainable with an intracavity adaptive optic that the exact adaptive compensation of linear phase-tilt mirror plane. In addition, the scalar-wave analysis has shown approach to understanding aberration effects on cavity mode metrical optics approximation provides a powerful analytical and decreasing aberration location with respect to the convex geometric. With its inherent simplicity, however, the geometrical optics approximation is strictly valid only in the limit of small aberration strengths. Its domain of validity should increase to larger aberration strengths as the cavity magnification and/or Fresnel number is increased (i.e., as the cavity mode properties become more geometric). With its inherent simplicity, however, the geometrical optics approximation provides a powerful analytical approach to understanding aberration effects on cavity mode properties. In particular, it was shown that the geometrical phase-weighting coefficient decreases monotonically with increasing cavity magnification, increasing aberration order, and decreasing aberration location with respect to the convex mirror plane. In addition, the scalar-wave analysis has shown that the exact adaptive compensation of linear phase-tilt aberrations is attainable with an intracavity adaptive optic element provided that no intracavity vignetting occurs. Furthermore, the mathematically defined ideal correction for arbitrary linear combinations of curvature of field, phase cylinder, and astigmatism results in a change in the transverse geometry of the cavity. However, an exact scalar-wave diffraction analysis of the ideal correction for higher-order aberration types is untenable. It is for these higher-order aberration types that the approximate geometrical analysis becomes more accurate and should yield results that are close to the optimum for moderate aberration strengths. These conclusions will be borne out in future papers in this series, which will present the results of a numerical study of the intracavity adaptive optics concept.

3. CONCLUSIONS
On the basis of the analysis of this paper, several general conclusions regarding the effects of linear intracavity phase aberrations on the mode-structure properties of a passive linear positive branch, confocal unstable resonator cavity may be made. First, the geometrical optics approximation is strictly valid only in the limit of small aberration strengths. Its domain of validity should increase to larger aberration strengths as the cavity magnification and/or Fresnel number is increased (i.e., as the cavity mode properties become more geometric). With its inherent simplicity, however, the geometrical optics approximation provides a powerful analytical approach to understanding aberration effects on cavity mode properties. In particular, it was shown that the geometrical phase-weighting coefficient decreases monotonically with increasing cavity magnification, increasing aberration order, and decreasing aberration location with respect to the convex mirror plane. In addition, the scalar-wave analysis has shown that the exact adaptive compensation of linear phase-tilt aberrations is attainable with an intracavity adaptive optic element provided that no intracavity vignetting occurs. Furthermore, the mathematically defined ideal correction for arbitrary linear combinations of curvature of field, phase cylinder, and astigmatism results in a change in the transverse geometry of the cavity. However, an exact scalar-wave diffraction analysis of the ideal correction for higher-order aberration types is untenable. It is for these higher-order aberration types that the approximate geometrical analysis becomes more accurate and should yield results that are close to the optimum for moderate aberration strengths. These conclusions will be borne out in future papers in this series, which will present the results of a numerical study of the intracavity adaptive optics concept.

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