Motivation

...Which is how we can determine the variance for binomial distribution!

Pretty great, right? Anybody see that coming?

The validity of Proposition 3 for L=4 is thus confirmed in a most elegant fashion!

Now, as you can see, this is a very complicated proof. You can't possibly track it without fully concentrating.

And yet it is perfectly obvious to me that most of you are either online or texting right now.

Which is puzzling because on a pro-rated basis, the lecture you're not listening to right now is costing you or your parents $75.

So I'd love to know -- what's the thinking here? Why are you so happy to receive nothing for your money?

MR. HARRIS?

"Go, everyone stop texting."
Magnetic Material Properties

Magnetic material properties originate in the motion of bound atomic electrons together with their intrinsic quantum mechanical spin.

- In most materials, bound electrons orbit the nucleus within specific energy bands. These energy bands are subdivided into orbitals which are typically evenly filled with electrons that pair together such that the paired orbits are orientationally alike but oppositely directed. In that case, the paired magnetic dipoles are oppositely directed and consequently cancel each other, resulting in a material exhibiting negligible magnetic properties.

- In those few materials that do exhibit macroscopic magnetic properties, the inner electron orbitals are not evenly filled so that those electrons are not completely paired. The resultant interaction between adjacent microscopic magnetic dipoles results in a coupling tending to align those magnetic moments, resulting in a material that exhibits magnetic properties.
Magnetic Material Properties

- If the magnetic moments tend to align themselves under the influence of an external magnetic field such that they tend to weaken the field, then the material is said to be **diamagnetic**.
- If the magnetic moments tend to align themselves with the external magnetic field such that they tend to strengthen the field, then the material is said to be **paramagnetic**.
- In a **ferromagnetic** material, the microscopic magnetic moments are highly aligned in the absence of an applied magnetic field. Because it is energetically favored, a ferromagnetic material is broken up into randomly oriented ferromagnetic domains where each domain is fully magnetized. The macro-magnetization properties of ferromagnetic materials is then a result of changes in the domain structure due to the application of a magnetic field, either through domain wall movement or through rotation of the domain magnetization, depending upon the applied magnetic field strength.
Electron spin alignment also plays a role in determining the magnetization properties of a material.

- If the spin alignment of neighboring atoms is parallel, then the material is **ferromagnetic**.
- If the favored spin alignment results in a net zero macroscopic magnetic moment, then the material is said to be **antiferromagnetic**.
- If the spin structure is comprised of both spin-up & spin-down components that results in a nonzero macroscopic magnetic moment, then the material is said to be **ferrimagnetic**.
In a perfect magnetic material, all electronic motion is confined to the atomic structure in the form of atomic currents and their moments are randomly oriented in the absence of an external applied magnetic field. They are then either paramagnetic or diamagnetic materials.

For a simple magnetic or magnetizable material, the dominant multipole moment of the atomic current distribution is the magnetic dipole, all higher-order moments being negligible by comparison. Let $m_j$ denote the microscopic magnetic dipole moment of the $j$th-type atom comprising the material. The macroscopic magnetic moment density or magnetization is then given by the spatial average

$$M(r) = \sum_j N_j \langle m_j(r) \rangle,$$

where $N_j$ is the average number density of $j$-type atoms in that microscopic region.
The \textit{macroscopic magnetic induction field vector} \( B(r) \) for the steady-state magnetic field is defined as the spatial average of the \textit{microscopic magnetic induction field vector} \( b(r) \) as

\[ B(r) \equiv \langle b(r) \rangle. \] (2)

There is then no distinction between the microscopic and macroscopic field quantities in vacuum. When the spatial-averaging procedure is applied to \textit{Gauss' law} (18.11) for the microscopic magnetic field, the same equation

\[ \nabla \cdot B(r) = 0 \] (3)

is found to hold for the macroscopic field vector. This result then implies that the macroscopic magnetic field may likewise be expressed in terms of a \textit{macroscopic vector potential} \( A(r) \) as

\[ B(r) = \nabla \times A(r). \] (4)
The spatial average of *Ampère’s law* (18.13) for the microscopic magnetic field yields
\[ \nabla \times \mathbf{B} = \mu_0 \langle \mathbf{j}(\mathbf{r}) \rangle, \quad (5) \]
so that the curl of the macroscopic magnetic induction vector is determined by the spatial average of the microscopic current density.

Just as for the spatial average of the microscopic charge density in a perfect dielectric, the spatial average of the microscopic current density in a perfect magnetic material deserves careful attention. Fortunately, the method of analysis for the present problem closely parallels that for a perfect dielectric, as given in Topic 11.
Macroscopic Magnetostatic Fields

The vector potential $A(r)$ and magnetic induction field $B(r)$ at points either interior or exterior to (but not on the surface of) the body of a perfect magnetic material are found to be given by

$$A(r) = \frac{\mu_0}{4\pi} \left[ \int_S \frac{M(r') \times \hat{n}}{r} \, d^2 r' + \int \int \int_V \frac{\nabla' \times M(r')}{r} \, d^3 r' \right],$$

$$B(r) = \frac{\mu_0}{4\pi} \left[ \int_S \frac{(M(r') \times \hat{n}) \times \hat{R}}{r} \, d^2 r' + \int \int \int_V \frac{(\nabla' \times M(r')) \times \hat{R}}{r} \, d^3 r' \right],$$

where $\hat{R} = (r - r')/|r - r'|$ and $r = |r - r'|$.

Notice that a free surface current density may be added to the numerator of the integrand in the above surface integral contributions and that a free volume current density may be added to the numerator of the integrand in the above volume integral contributions in order to account for any externally supplied current sources.
Macroscopic Magnetostatic Fields

For a simple magnetic material, the spatial average of the micro-current density is thus seen to be given by

$$\langle j(r) \rangle = J(r) + \nabla \times M(r), \quad (6)$$

where $J$ denotes the \textit{macroscopic current density} in the magnetic material and where $M$ is the \textit{macroscopic magnetization}. The \textit{magnetization current density} $J_m$ is then defined as the curl of the magnetization as

$$J_m(r) \equiv \nabla \times M(r) \quad (7)$$

with associated \textit{magnetization surface current density}

$$J_{sm}(r) \equiv M(r) \times \hat{n}. \quad (8)$$
With substitution from (6), the spatial average (5) of *Ampère’s law* becomes

\[ \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) \]  

(9)

where

\[ \mathbf{H}(\mathbf{r}) \equiv \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r}) \quad (A/m) \]  

(10)

is the *magnetic intensity vector* for a simple magnetic material.

Because \( \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{H}(\mathbf{r}) + \mu_0 \mathbf{M}(\mathbf{r}) \), the \( \mathbf{B}(\mathbf{r}) \)-field inside a simple magnetic material is seen to be given by the sum of two vector fields:

- the field \( \mu_0 \mathbf{H}(\mathbf{r}) \) associated with the spatially averaged atomic current density plus any externally supplied current, where \( \nabla \times (\mu_0 \mathbf{H}(\mathbf{r})) = \mu_0 \mathbf{J}(\mathbf{r}) \), and
- the field \( \mu_0 \mathbf{M}(\mathbf{r}) \) associated with the magnetization current density, where \( \nabla \times (\mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_M(\mathbf{r}) \).
For a simple magnetizable material, the macroscopic magnetization $M(r)$ is linearly related to and in the same direction as the macroscopic magnetic intensity vector $H(r)$ at that point, so that

$$M(r) \equiv \chi_m H(r), \quad (11)$$

where $\chi_m$ is the magnetic susceptibility. Substitution in Eq. (10) then gives

$$B(r) = \mu_0 (1 + \chi_m) H(r). \quad (12)$$

If $\chi_m$ is positive, the magnetizable material is said to be paramagnetic and the magnetic induction field is strengthened by its interaction with the material. If $\chi_m$ is negative, the magnetizable material is said to be diamagnetic and the magnetic induction field is weakened by its interaction with the material. In general, the magnitude of $\chi_m$ is quite small for both paramagnetic and diamagnetic materials, so that

$$|\chi_m| \ll 1. \quad (13)$$
The \textit{magnetic permeability} $\mu$ of the magnetizable medium is then defined as

$$\mu \equiv \mu_0 (1 + \chi_m), \quad \text{(14)}$$

and the \textit{relative permeability} is then given by $\mu_r = \mu/\mu_0$, where $\mu$ is real-valued in the static case. The relation (12) between the macroscopic magnetic induction and intensity vectors then becomes

$$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r}) \quad \text{(15)}$$

for simple (perfect) magnetic materials.

This linear relationship (15) does not apply for either ferromagnetic, antiferromagnetic, or ferrimagnetic materials as they exhibit \textit{hysteresis}, a rate-independent branching nonlinearity relating the $\mathbf{B}$- and $\mathbf{H}$-fields.
Boundary conditions for the magnetostatic field vectors across an interface $S$ separating two simple magnetic materials with permeabilities $\mu_1$ and $\mu_2$ may be obtained by direct application of the integral form of Gauss’ law to a simple closed surface with identical faces on opposite sides of $S$ and Ampère’s law to a simple closed circuit with identical segments on opposite sides of $S$ with the results

\[
\hat{n} \cdot \left( B_2(r) - B_1(r) \right) = 0, \quad r \in S, \\
\hat{n} \times \left( H_2(r) - H_1(r) \right) = J_s(r), \quad r \in S,
\]

where $\hat{n}$ is the unit normal to the surface at the point $r$, directed from medium 1 into medium 2. Notice that, unless one of the materials is a superconductor\(^2\), the surface current density $J_s(r) = 0$.

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\(^2\)A superconductor is a material with $\sigma_0 \to \infty$ so that $E = 0$ in its interior and which also completely excludes magnetic flux in its interior so that $B = 0$ there.