Motivation
The **instantaneous current** is defined as the time rate of change of charge transfer viz.

\[
i(t) \equiv \frac{dq(t)}{dt}. \quad (A)
\]

If charge with density \( \varrho(r, t) \) is moving with velocity \( \mathbf{v}(r, t) \), the corresponding **convective current density** is

\[
\mathbf{J}(r, t) = \varrho(r, t)\mathbf{v}(r, t). \quad (A/m^2)
\]

Notice that if \( \varrho \) is positive, then \( \mathbf{J} \) is in the same direction as \( \mathbf{v} \), but if \( \varrho \) is negative, then \( \mathbf{J} \) is in the opposite direction to \( \mathbf{v} \).
Let $da$ be an infinitesimal element of a regular surface with unit normal vector $\hat{n}$, as illustrated.

If the current density $\mathbf{J}(r, t)$ is evaluated at an interior point of $da$, then the quantity $\mathbf{J} \cdot \hat{n} da$ is the rate at which charge flows across $da$ either into the region of space that $\hat{n}$ is directed if $\varrho$ is positive, or into the region of space that $-\hat{n}$ is directed if $\varrho$ is negative.
Conservation of Charge

The net flow of charge per unit area per unit time across an arbitrarily oriented surface element $\hat{n} \, da$ is then given by

$$\mathcal{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) \cdot \hat{n} \, da.$$  \hfill (3)

The total current flowing across a regular surface $S$ into the region of space that the unit surface normal $\hat{n}$ to $S$ is directed is then given by the surface integral

$$\iota(t) = \int \int_S \mathbf{J}(\mathbf{r}, t) \cdot \hat{n} \, da.$$  \hfill (4)

For a closed surface $S$, conservation of charge requires that

$$\oint_S \mathbf{J}(\mathbf{r}, t) \cdot \hat{n} \, da = -\frac{d}{dt} \int \int \int_V \varrho(\mathbf{r}, t) \, d^3r$$  \hfill (5)

where $V$ is the volume enclosed by $S$. 
Equation of Continuity

If the surface $S$ is fixed in space, Leibniz’ rule yields

$$\oint_S \mathbf{J}(\mathbf{r}, t) \cdot \hat{n} da = - \iiint_V \frac{\partial \rho(\mathbf{r}, t)}{\partial t} d^3 r.$$

Application of the divergence theorem to this result then gives

$$\iiint_V \left( \nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} \right) d^3 r = 0.$$

Because this integral must vanish for an arbitrary region $V$, it is then necessary that the integrand itself be identically zero at all points of space, resulting in the **Equation of Continuity**

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0 \quad (6)$$
Consider a spatially homogeneous, isotropic medium with dielectric permittivity \( \epsilon \) and conductivity \( \sigma \) containing an initial volume density of charge \( \varrho_0(\mathbf{r}) = \varrho_0(x, y, z) \) at time \( t = 0 \).

If, at time \( t = 0 \), this charged body is instantaneously isolated from the applied electrostatic field maintaining this charge distribution, it will then evolve toward a new equilibrium state where there is no excess charge in its interior.

At all times, the equation of continuity

\[
\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \varrho(\mathbf{r}, t)}{\partial t} = 0
\]

is satisfied, which, with Ohm’s law \( \mathbf{J}(\mathbf{r}, t) = \sigma \mathbf{E}(\mathbf{r}, t) \), becomes

\[
\sigma \nabla \cdot \mathbf{E}(\mathbf{r}, t) + \frac{\partial \varrho(\mathbf{r}, t)}{\partial t} = 0.
\]
From the time-domain form of Gauss’ law, \( \nabla \cdot \mathbf{E}(r, t) = \varrho(r, t)/\epsilon \), this equation becomes

\[
\frac{\partial \varrho(r, t)}{\partial t} + \frac{\sigma}{\epsilon} \varrho(r, t) = 0,
\]

with solution

\[
\varrho(r, t) = \varrho_0(r) e^{-\frac{\sigma}{\epsilon} t}
\]

satisfying the initial condition \( \varrho(r, 0) = \varrho_0 \).

The equilibrium state of vanishing internal charge density is then approached exponentially with time constant

\[
t_c \equiv \frac{\epsilon}{\sigma},
\]

which is also referred to as the relaxation time of the medium.
The relaxation time $t_c$ of a material is a measure of how fast it approaches electrostatic equilibrium under a given set of conditions.

For a perfect conductor ($\sigma = 0$), the relaxation time is zero and equilibrium is attained instantaneously.

A material will exhibit conductor-like behavior in a given applied electric field if its characteristic relaxation time $t_c$ is much shorter than the time describing a change in the applied electric field.
Lightning (Donner und Blitzen)

”Thunder is good. Thunder is impressive. But it’s the lightning that does the work,” Samuel Clemens.

Cloud to ground lightning is the most common form of lightning. In an electrical storm, the storm clouds become capacitively charged with the upper portion of the cloud positive and the lower portion negative.
How the cloud acquires this charge is still not agreed upon within the scientific community, but the following description provides one plausible explanation: J. Zavisa [http://science.howstuffworks.com]

- In the process of the water cycle, moisture can accumulate in the atmosphere forming clouds containing millions upon millions of water droplets and ice suspended in the air. As the process of evaporation and condensation continues, these droplets collide with moisture that is in the process of condensing as it rises. The rising moisture may also collide with ice or sleet that is either in the process of falling to the earth or is located in the lower portion of the cloud.
- The importance of these collisions is that electrons are knocked off of the rising moisture, thus creating a charge separation. The newly knocked-off electrons gather at the lower portion of the cloud, giving it a negative charge. The rising moisture that has just lost an electron then carries a positive charge to the top of the cloud.
• Beyond the collisions, freezing plays an important role. As the rising moisture encounters colder temperatures in the upper cloud regions and begins to freeze, the frozen portion becomes negatively charged and the unfrozen droplets become positively charged. At this point, rising air currents have the ability to remove the positively charged droplets from the ice and carry them to the top of the cloud. The remaining frozen portion would then likely fall to the lower portion of the cloud or continue on to the ground.
Lightning (Electric Field Generation)

The combined effects of collisions and freezing provide a simple explanation of how a cloud may acquire the extreme charge separation that is required for a lightning strike. This charge separation produces an electric field that is negative in the lower region and positive in the upper region, the strength or intensity being directly related to the amount of charge separation in the cloud.

As the charges at the top and bottom of the cloud increase, the electric field becomes so intense that the electrons at the earth’s surface are repelled deeper into the earth by the strong negative charge at the lower portion of the cloud. This repulsion of electrons causes the earth’s surface to acquire a strong positive charge.

All that is needed now is a conductive path for the negative cloud bottom to contact the positive earth surface. The strong electric field, being somewhat self-sufficient, creates this path.
Lightning (Dielectric Breakdown)

The strong electric field causes the air around the cloud to break down (dielectric breakdown), allowing current to flow in an attempt to neutralize the charge separation. Simply stated, the air breakdown creates a path that short-circuits the cloud/earth.
Lightning (Dielectric Breakdown)

When the electric field strength becomes on the order of millions of volts per meter, conditions for dielectric breakdown in air are satisfied. The electric field causes the surrounding air to ionize, separating it into positive ions and electrons. The electrons are now free to move much more easily than they could before the separation. This ionized air (also known as a plasma) is now much more conductive than the previous non-ionized air.

The maximum electric field intensity that a dielectric can withstand without breakdown is the dielectric strength $E_{bd}$ of the material\(^1\).

- Dielectric Strength of Air: $E_{bd} = 3 \times 10^6 \text{V/m}$
- Dielectric Strength of Water: $E_{bd} = 1.5 \times 10^7 \text{V/m}$

This ionization process may be viewed as ”burning a path” through the air for the lightning to follow. After the ionization process, the path between the cloud and the earth begins to form.

Lightning (Step Leaders)

Once the ionization process begins and a plasma forms, a path is not created almost instantaneously. In fact, there are usually many separate paths of ionized air stemming from the cloud. These paths are typically referred to as “step leaders”.

Step leaders are paths of ionized air stemming from the cloud, illustrated in the above NASA photograph.
The step leaders propagate toward the earth in stages, which do not have to result in a straight line to the earth. The air may not ionize equally in all directions. Dust or impurities in the air may cause the air to break down more easily in one direction, giving a better chance that the step leader will reach the earth faster in that direction.

In addition, the shape of the electric field can greatly affect the ionization path. This shape depends on the location of the charged particles, which in this case are located at the bottom of the cloud and the earth’s surface. If the cloud is parallel to the earth’s surface, and the area is small enough that the curvature of the earth is negligible, the two charge locations will behave as two charged parallel plates. The lines of force (electric flux) generated by the charge separation will be perpendicular to the cloud and earth (bc’s). These flux lines always radiate perpendicularly from the charge surface before moving toward the oppositely charged location.
It is then clear that there are various factors that affect the direction of the step leader. The lines of force (flux lines) may not follow the shortest distance, as the shortest distance does not always represent the path of least resistance.

So now we have an electrically charged cloud with ever-growing step leaders stretching out toward the earth in stages. These leaders are faintly illuminated in a purplish glow and may sprout other leaders in areas where the original leaders bend or turn. Once begun, the leader will remain until the current flows, regardless of whether or not it is the leader that reaches the ground first.
Each step leader basically has two possibilities:

1. continue to grow in stages of growing plasma, or
2. wait patiently in its present plasma condition until another leader hits a target.

The leader that reaches the earth first reaps the rewards of the journey by providing a conductive path between the cloud and the earth. This leader is not the lightning strike; it only maps out the course that the strike will follow. The strike is the sudden, massive, flow of electrical current moving from the cloud to the ground.
Lightning (Positive Streamers)

As the step leaders approach the earth, objects on the surface begin responding to the strong electric field. The objects reach out to the cloud by growing positive streamers. These streamers also have a purplish color and appear to be more prominent on sharp edges.

The human body can and does produce these positive streamers when subjected to a strong electric field such as that of a storm cloud. In actuality, anything on the surface of the earth has the potential to send a streamer.

Once produced, the streamers do not continue to grow toward the clouds; bridging the gap is the job of the step leaders as they stage their way down. The streamers wait patiently, stretching upward as the step leaders approach.
Lightning (Positive Streamers)
Next to occur is the actual meeting of a step leader and a streamer. The streamer that the step leader reaches is not necessarily the closest streamer to the cloud. After the step leader and the streamer meet, the ionized air (plasma) has completed its journey to the earth, leaving a conductive path from the cloud to the earth. With this path complete, current flows between the earth and the cloud. This discharge of current is nature’s way of trying to neutralize the charge separation. The flash we see when this discharge occurs is not the strike – it is the local effects of the strike.

When a leader and a streamer meet and the current flows (the strike), the air around the strike becomes extremely hot. So hot that it actually explodes because the heat causes the air to expand so rapidly. The explosion is soon followed by thunder which is just the shock-wave radiating away from the strike path.