Fire is invented.
The Concept of a Perfect Dielectric

In a perfect dielectric medium, charged particles are bound either in atomic or molecular configurations.

When an external electric field $\mathbf{E}$ is applied, positive and negative charges bound in each molecule are displaced in opposite directions and the molecular charge density of each molecule is distorted. Positive charge is displaced in the direction of $\mathbf{E}$ and negative charge in the opposite direction so that the induced molecular dipole moment is in the same direction as $\mathbf{E}$. Each molecule then produces an average electric field that is in a direction opposite to $\mathbf{E}$. 
For a *simple dielectric*, the dominant induced multipole is the dipole, all higher-order multipoles being negligible by comparison. The dielectric is then *polarized* by the external electric field with its molecules possessing *induced dipole moments*.

In a *nonpolar dielectric*, the molecules have zero permanent dipole moments. In the absence of an applied electric field, the dipole (and higher-order multipole) moments are all zero.

In a *polar dielectric*, the molecules possess a nonzero permanent dipole moment. In the absence of an applied electrostatic field, the molecular dipole moments are randomly oriented so that the spatially-averaged dipole (and higher-order multipole) moments are again all zero.
Application of an external electric field in such a simple dielectric then produces a *macroscopic electric polarization density* \( \mathbf{P}(\mathbf{r}) \) given by the spatial average of the microscopic dipole moment as

\[
\mathbf{P}(\mathbf{r}) = \sum_j N_j \langle \mathbf{p}_j(\mathbf{r}) \rangle, \tag{1}
\]

where \( \mathbf{p}_j(\mathbf{r}) \) is the *microscopic dipole moment* of the \( j \)th-type of molecule comprising the dielectric, \( \langle \mathbf{p}_j(\mathbf{r}) \rangle \) is the spatial average of this microscopic dipole moment taken over a macroscopically small but microscopically large region centered at the point \( \mathbf{r} \), and where \( N_j \) is the average number density of \( j \)-type molecules in that region.

Because the net charge in a perfect dielectric is zero, the *macroscopic dipole moment density* \( \mathbf{P}(\mathbf{r}) \) is independent of the choice of origin.\(^1\)

\(^1\)See the discussion following Eq. (10.10).
Macroscopic Electrostatics

The connection between microscopic and macroscopic field quantities is obtained through the same spatial-averaging process. The *macroscopic electric field vector* \( \mathbf{E}(\mathbf{r}) \) is defined as the spatial average of the *microscopic electric field vector* \( \mathbf{e}(\mathbf{r}) \) as

\[
\mathbf{E}(\mathbf{r}) \equiv \langle \mathbf{e}(\mathbf{r}) \rangle. \tag{2}
\]

There is then no distinction between microscopic and macroscopic fields in vacuum. When the spatial-averaging procedure is applied to *Faraday’s law* (1.24) for the microscopic electrostatic field, the same equation results, viz.

\[
\nabla \times \mathbf{E}(\mathbf{r}) = 0. \tag{3}
\]

This result implies that the macroscopic field may likewise be expressed in terms of a *macroscopic scalar potential* \( V(\mathbf{r}) \) as

\[
\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}). \tag{4}
\]
The spatial average of *Gauss’ law* (1.20) for the microscopic electrostatic field yields

\[ \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} \langle \rho(\mathbf{r}) \rangle, \]

so that the divergence of the macroscopic electrostatic field vector is determined by the spatial average of the microscopic charge density in the dielectric. The proper description of this quantity deserves careful attention.
Case I: Electrostatic Field produced by a Dielectric Body with Dipole Moment Density $\mathbf{P}(\mathbf{r})$ at an Exterior Field Point.

From Eq. (9.6), the differential element of potential $dV(\mathbf{r})$ produced at an exterior point $\mathbf{r}$ in vacuum due to the differential dipole source $\mathbf{P}(\mathbf{r}')d^3r'$ at an interior point $\mathbf{r}'$ is given by

$$(4\pi\varepsilon_0)dV(\mathbf{r}) = (\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{1}}_r / r^2)d^3r' = [\mathbf{P}(\mathbf{r}') \cdot \nabla'(1/r)]d^3r',$$

where $\hat{\mathbf{1}}_r$ is the unit vector directed from the source point $\mathbf{r}'$ to the field point $\mathbf{r}$, separated by the distance

$$r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2},$$

and where $\nabla'$ operates on the primed (source) coordinates.
Macroscopic Electrostatics

The total potential at the exterior point \( \mathbf{r} \) in vacuum is then given by

\[
V(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \int \int \int_{\mathcal{V}} \mathbf{P} \cdot \nabla' \left( \frac{1}{r} \right) \, d^3r' \\
&= \frac{1}{4\pi\varepsilon_0} \left[ \int \int \int_{\mathcal{V}} \nabla' \cdot \left( \frac{\mathbf{P}}{r} \right) \, d^3r' - \int \int \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{P}}{r} \, d^3r' \right] \\
&= \frac{1}{4\pi\varepsilon_0} \left[ \oint_{\mathcal{S}} \mathbf{P} \cdot \hat{n} \, d^2r' - \int \int \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{P}}{r} \, d^3r' \right],
\]

where \( \mathcal{S} \) is the boundary surface to the dielectric region \( \mathcal{V} \) with outward unit normal vector \( \hat{n} \).

Comparison of this result with Coulomb’s law (1.10) shows that this external field is produced by a **surface polarization charge density**

\[
\varrho_{sb}(\mathbf{r}) \equiv \mathbf{P}(\mathbf{r}) \cdot \hat{n}, \mathbf{r} \in \mathcal{S},
\]

and a **volume polarization charge density**

\[
\varrho_{b}(\mathbf{r}) \equiv -\nabla \cdot \mathbf{P}(\mathbf{r}), \mathbf{r} \in \mathcal{V},
\]

so that

\[
V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[ \oint_{\mathcal{S}} \frac{\varrho_{sb}(\mathbf{r}')}{r} \, d^2r' + \int \int \int_{\mathcal{V}} \frac{\varrho_{b}(\mathbf{r}')}{r} \, d^3r' \right]. \tag{6}
\]
Case II: Electrostatic Field produced by a Dielectric Body with Dipole Moment Density $\mathbf{P}(\mathbf{r})$ at an Interior Field Point.

Consider a fixed point $\zeta$ inside the dielectric body $\mathcal{V}$ such that a sphere of radius $R$ centered at $\zeta$ can be constructed with surface $S_\zeta$ lying entirely within $\mathcal{V}$, dividing the dielectric into two regions: $\mathcal{V}'$ inside $S_\zeta$ and $\mathcal{V}''$ outside $S_\zeta$ but inside $S$.

The radius $R$ is chosen sufficiently small that the enclosed volume $\mathcal{V}'$ is macroscopically small, in which case the macroscopic electric field vector $\mathbf{E}(\mathbf{r})$, dipole moment density $\mathbf{P}(\mathbf{r})$, and polarization charge density $\varrho_b = -\nabla \cdot \mathbf{P}$ do not vary appreciably in $\mathcal{V}'$. 
The potential at $\zeta$ due to the dipole distribution in the exterior region $\mathcal{V}''$ is given by Eq. (6) with the surface integral taken over both $S$ and $S_\zeta$ and the volume integral taken over $\mathcal{V}''$. The corresponding electric field intensity is then given by

$$E''(\zeta) = \frac{1}{4\pi \varepsilon_0} \left[ \int_S \hat{r} \frac{Q_{sb}(\mathbf{r}')}{r^2} d^2 r' + \int_{S_\zeta} \hat{r} \frac{Q_{sb}(\mathbf{r}')}{r^2} d^2 r' + \int_{\mathcal{V}''} \hat{r} \frac{P_b(\mathbf{r}')}{r^2} d^3 r' \right],$$

where $\hat{r}$ is the unit vector from the dipole source to the field point $\zeta$ and $r$ is the distance between these two points.
The average electric field intensity $E'(\zeta)$ due to the near dipoles enclosed by $S_\zeta$ is given by the summation over $j$ of the product of the molecular dipole field $-\langle p_j \rangle/(4\pi \epsilon_0 R^3)$, where $\langle p_j \rangle$ denotes the average dipole moment of the $j$-type molecule in $S_\zeta$, times the number density $N_j$ of $j$-type molecules times the volume $(4/3)\pi R^3$ enclosed by $S_\zeta$, so that

$$E'(\zeta) = -\frac{4}{3} \pi R^3 \sum_j N_j \langle p_j(r) \rangle \frac{\langle p_j \rangle}{4\pi \epsilon_0 R^3} = -\frac{P}{3\epsilon_0},$$

where $P = \sum_j N_j \langle p_j(r) \rangle$, as defined in Eq. (1).
The integral over spherical surface $S_\zeta$ may be evaluated as follows:

Construct an axis in the direction $\hat{1}_P$ parallel to $P$ and let $\theta$ be the angle between $\hat{1}_P$ and the line extending from a point on $S_\zeta$ to the point $\zeta$, so that $\varrho_{sb} = P \cdot \hat{n'} = -P \cos \theta$, where $\hat{n'}$ is the unit normal to $S_\zeta$ directed inwards toward the center $\zeta$. 
Then
\[ \int_{S_{\zeta}} \hat{r} \frac{Q_{sb}(r')}{r^2} d^2r' = 2\pi P \int_{0}^{\pi} \cos^2 \theta \sin \theta d\theta = \frac{4}{3} \pi P, \]
since \( \hat{r} = -\hat{1}_{\zeta} \cos \theta \) for this geometry. This term then cancels the near dipole contribution \( E' (\zeta) = -P/3\epsilon_0 \) given above.

The total electrostatic field intensity produced at an interior point of a polarized dielectric is then given by

\[ E(r) = \frac{1}{4\pi \epsilon_0} \left[ \int_{S} \hat{r} \frac{Q_{sb}(r')}{r^2} d^2r' + \int \int \int_{\mathcal{V}''} \hat{r} \frac{Q_{b}(r')}{r^2} d^3r' \right], \]

where the integration over \( \mathcal{V}'' \) has been extended to the entire volume \( \mathcal{V} \) of the dielectric body by adding the contribution from the spherical region \( \mathcal{V}' \) which vanishes as it describes the electrostatic field at the center of a uniform spherical charge distribution.
Taken together, these results show that the electrostatic potential and field both inside and outside a simple dielectric with free charge are given by

\[
V(r) = \frac{1}{4\pi \epsilon_0} \left[ \int_S \frac{\varrho_s(r') + \varrho_{sb}(r')}{r} d^2 r' + \int \int \int_V \frac{\varrho(r') + \varrho_b(r')}{r} d^3 r' \right],
\]

\[
E(r) = \frac{1}{4\pi \epsilon_0} \left[ \int_S \hat{r} \frac{\varrho_s(r') + \varrho_{sb}(r')}{r^2} d^2 r' + \int \int \int_V \hat{r} \frac{\varrho(r') + \varrho_b(r')}{r^2} d^3 r' \right],
\]

where

- \( \varrho_s \) is the \textit{free surface charge density} on \( S \),
- \( \varrho \) is the \textit{free volume charge density} in \( V \),

and where

- \( \varrho_{sb}(r) \equiv \mathbf{P}(r) \cdot \hat{n}, r \in S \) is the \textit{surface polarization charge density},
- \( \varrho_b(r) \equiv -\nabla \cdot \mathbf{P}(r), r \in V \) is the \textit{volume polarization charge density}. 
For a simple dielectric, the spatial average of the microscopic charge density is

$$\langle \rho(r) \rangle = \rho(r) - \nabla \cdot P(r),$$

(7)

where $\rho$ is the \textit{macroscopic charge density} in the dielectric and $P$ is the macroscopic polarization density defined in Eq. (1). The presence of the divergence of $P(r)$ in this spatial average accounts for any spatial nonuniformity in this vector field and is referred to as the \textit{polarization} or \textit{bound charge density} $\rho_b$, where

$$\rho_b(r) \equiv -\nabla \cdot P(r)$$

(8)

for all points $r$ interior to the dielectric body. In addition, associated with this macroscopic dipole moment density is the \textit{surface polarization charge density}

$$\rho_{sb}(r) \equiv P(r) \cdot \hat{n}$$

(9)

for all points on the surface of the dielectric body with outward unit normal vector $\hat{n}$. 

\textbf{Macroscopic Electrostatics: Simple Dielectrics}
The macroscopic differential form of Gauss’ law is then given by

\[ \nabla \cdot \mathbf{D}(\mathbf{r}) = \varrho(\mathbf{r}) \]  \hspace{1cm} (10)

with associated integral form

\[ \oint_S \mathbf{D}(\mathbf{r}) \cdot \hat{n} d^2r = \iiint_V \varrho(\mathbf{r}) d^3r \]  \hspace{1cm} (11)

where \( V \) is the region enclosed by the simply-connected surface \( S \).

This macroscopic integral form of Gauss’ law states that the flux of the electric displacement vector \( \mathbf{D}(\mathbf{r}) \) passing through any simply-connected, closed surface \( S \) is equal to the total charge enclosed by the surface. This is usually just the free-charge because the spatially-averaged molecular (bound) charge is typically zero.
The macroscopic electric displacement vector for a simple dielectric is given by

\[ D(\mathbf{r}) = \epsilon_0 E(\mathbf{r}) + P(\mathbf{r}), \]  

so that

\[ E(\mathbf{r}) = \frac{1}{\epsilon_0} D(\mathbf{r}) - \frac{1}{\epsilon_0} P(\mathbf{r}). \]  

The macroscopic electrostatic field intensity \( E(\mathbf{r}) \) inside a simple dielectric is then given by the sum of two vector fields:

1. The field \( D(\mathbf{r})/\epsilon_0 \) associated with the free charge through

\[ \nabla \cdot D(\mathbf{r}) = \rho(\mathbf{r}) \rightarrow \rho_f(\mathbf{r}) \]  

2. The field \( -P(\mathbf{r})/\epsilon_0 \) associated with the bound charge through

\[ \nabla \cdot (-P(\mathbf{r})/\epsilon_0) = \frac{\rho_b(\mathbf{r})}{\epsilon_0} \]  

The \( D \)-field lines then begin or end on free charges, whereas the \( E \)-field lines begin or end on either free or bound charges.
The macroscopic polarization density \( \mathbf{P}(\mathbf{r}) \) in a simple, isotropic, linear dielectric is linearly related to the macroscopic electrostatic field intensity \( \mathbf{E}(\mathbf{r}) \) as

\[
\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r}) \quad (16)
\]

where \( \chi_e \) is the static electric susceptibility of the material.

With this substitution, Eq. (12) for the electric displacement vector in a simple dielectric becomes

\[
\mathbf{D}(\mathbf{r}) = \varepsilon_0 \left(1 + \chi_e\right) \mathbf{E}(\mathbf{r})
= \varepsilon_0 \varepsilon_r \mathbf{E}(\mathbf{r}) = \varepsilon \mathbf{E}(\mathbf{r}),
\quad (17)
\]

where \( \varepsilon \) is the static dielectric permittivity of the material, given by

\[
\varepsilon \equiv \varepsilon_0 \left(1 + \chi_e\right) \quad (18)
\]

in \( F/m \), and where \( \varepsilon_r \equiv \varepsilon / \varepsilon_0 = 1 + \chi_e \) is the relative permittivity. In vacuum, \( \chi_e = 0 \), \( \varepsilon = \varepsilon_0 \), and \( \varepsilon_r = 1 \).
From Eqs. (12) and (17), the macroscopic polarization density may be expressed as

\[ P(r) = D(r) - \epsilon_0 E(r) = \frac{\epsilon_r - 1}{\epsilon_r} D(r). \]

Because \( \nabla \cdot P = -\rho_b \) and \( \nabla \cdot D = \rho_f \) in a simple dielectric, then the divergence of the above relation gives

\[ \rho_b(r) = \left( \frac{1}{\epsilon_r} - 1 \right) \rho_f(r). \] (19)

The total macroscopic charge density in a simple dielectric is then

\[ \rho(r) = \rho_b(r) + \rho_f(r) = \frac{\rho_f(r)}{\epsilon_r}. \] (20)

which is smaller than the free charge density because \( \rho_b \) cancels part of \( \rho_f \). Hence, \( \rho_b(r) = 0 \) at any point where \( \rho_f(r) = 0 \). Typically, \( \rho_f = 0 \) throughout a dielectric so that \( \nabla \cdot P = 0 \) and \( \rho_b \) is situated entirely on the surface of the dielectric.
Macroscopic Electrostatics

Macroscopic electrostatics are described by the vector differential relations

\[ \nabla \cdot \mathbf{D}(\mathbf{r}) = \varrho(\mathbf{r}) \iff \oint_S \mathbf{D}(\mathbf{r}) \cdot \hat{n} d^2r = \iiint_V \varrho(\mathbf{r}) d^3r \]

\[ \nabla \times \mathbf{E}(\mathbf{r}) = 0 \iff \oint_C \mathbf{E}(\mathbf{r}) \cdot d\ell = 0 \]

The irrotational character of \( \mathbf{E}(\mathbf{r}) \) then implies that

\[ \mathbf{E}(\mathbf{r}) = -\nabla \varphi(\mathbf{r}). \]

If \( \epsilon \) is not a function of position in the dielectric (i.e., the material is spatially homogeneous), then Gauss’ law becomes

\[ \nabla \cdot \mathbf{E}(\mathbf{r}) = \varrho(\mathbf{r}) / \epsilon \] (21)

and the macroscopic electrostatic potential \( \varphi(\mathbf{r}) \) satisfies Poisson’s equation

\[ \nabla^2 \varphi(\mathbf{r}) = -\varrho(\mathbf{r}) / \epsilon \] (22)