Average Electrostatic Potential over a Spherical Surface

EE 141 Lecture Notes
Topic 8

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Consider a spherical surface $S$ of radius $a > 0$ carrying a uniform surface charge density $\varphi_s$ with total charge

$$Q = 4\pi a^2 \varphi_s.$$

Because of the spherical symmetry of the surface charge distribution, the electrostatic field vector is radially directed from the center $O$ of the sphere and is a function of the radial distance $R$ alone, so that

$$\mathbf{E}(r) = \hat{\mathbf{r}} R E(R).$$
By Gauss’ law, the electric field intensity at an observation point $P$ a distance $R > a$ from the center $O$ of the sphere is given by

$$E(R) = \frac{Q}{4\pi \epsilon_0 R^2},$$

and the absolute potential is

$$V(R) = \frac{Q}{4\pi \epsilon_0 R}.$$  

The potential is also given by Coulomb’s law [Eq. (4.5)] as

$$V(R) = \oint_S \frac{\rho_s}{4\pi \epsilon_0 r} \, d\alpha$$

where $\rho_s = Q/4\pi a^2$. 
Equating these two expressions for $V(R)$ when $R > a$ yields

$$\frac{Q}{4\pi\varepsilon_0 R} = \frac{Q}{4\pi a^2} \oint_S \frac{da}{4\pi\varepsilon_0 r}, \quad (1)$$

which then results in the geometrical identity

$$\frac{1}{R} = \frac{1}{4\pi a^2} \oint_S \frac{da}{r}, \quad R > a \quad (2)$$

The average value of $\frac{1}{r}$ taken over a spherical surface $S$, where $r$ is the distance from a point on the surface $S$ to an exterior point $P$, is equal to $\frac{1}{R}$, where $R$ is the distance from the center $O$ of the sphere $S$ to $P$. 
If one now removes the surface charge density \( \rho_s = Q/4\pi a^2 \) and places a point charge \( Q \) at the exterior point \( P \), then the potential at the center \( O \) of the sphere is given by the left-hand side of Eq. (1). The right-hand side of this equation is then just the average potential on the spherical surface. Hence:

The average potential over a spherical surface due to a point charge situated outside is equal to the potential at the center of the sphere.

Because of the principle of superposition, one then has the general result:

**Mean-Value Theorem:** The average potential over any spherical surface is equal to the potential at the center of the sphere if there are no charges inside the sphere.
As a corollary of this result, one has that:

**It is impossible to have a potential maximum or minimum in a charge-free region.**

In order to show this, suppose that there is a potential maximum (or minimum) at some point $P'$ in a charge-free region of space. The average potential over some sphere centered on $P'$ must then be lower (or higher) than the potential at $P'$, which contradicts the mean-value theorem.

Notice that this corollary is also a consequence of Laplace’s equation [see the discussion following Eq. (4.6)].
Consider again a spherical surface $S$ of radius $a$ carrying a uniform surface charge density $\rho_s = Q/4\pi a^2$ with total charge $Q$.

Application of Gauss’ law [see Eq. (1.16)] to any concentric spherical surface $S'$ of radius $R < a$ shows that, at any point $P$ inside $S$, the electrostatic field intensity is zero because there isn’t any enclosed charge.
The electrostatic potential \( V(R) \) at any point \( P \) interior to \( S \) must then be equal to the potential at the surface, so that

\[
V(R) = \frac{Q}{4\pi\varepsilon_0 a} = \int_S \frac{\varrho_s}{4\pi\varepsilon_0 r} \, da = \frac{Q}{4\pi a^2} \int_S \frac{da}{4\pi\varepsilon_0 r},
\]

which results in the geometrical identity

\[
\frac{1}{a} = \frac{1}{4\pi a^2} \int_S \frac{da}{r}, \quad R < a
\]
If one now removes the surface charge density $\varrho_s = Q/4\pi a^2$ and places a point charge $Q$ at the interior point $P$, then the final expression in Eq. (3) is seen to be the average potential taken over the spherical surface $S$. Hence:

The average potential taken over a spherical surface of radius $a$ containing a point charge $Q$ in its interior is equal to $Q/4\pi \epsilon_0 a$ irrespective of the position of the charge $Q$ inside the surface.

Because of the principle of superposition, one then has the general result:

The average electrostatic potential taken over any spherical surface is equal to $\frac{Q}{4\pi \epsilon_0 a}$, where $a$ is the radius of the sphere and $Q$ is the total enclosed charge, provided that there is no charge outside the sphere.