Motivation (Part 1)

AAGHH, I CAN'T BELIEVE WE WERE ASSIGNED TO DO A REPORT TOGETHER.

ALL I CAN SAY IS YOU'D BETTER DO A GREAT JOB! I DON'T WANT TO FLUNK JUST BECAUSE I WAS ASSIGNED A DOOFUS FOR A PARTNER.

A DOOFUS? WHO TAKES HER SANDWICHES APART AND EATS EACH INGREDIENT SEPARATELY?

WHAT'S WRONG WITH THAT??

IT CERTIFIES YOU AS A GRADE "A" NIMROD.

IT DOES NOT!

OK, LOOK. WE'VE GOT TO DO THIS DUMB PROJECT TOGETHER SO WE MIGHT AS WELL GET IT OVER WITH. WHAT ARE WE SUPPOSED TO BE DOING?

WEREN'T YOU EVEN PAYING ATTENTION?? WHAT WOULD YOU DO IF I WASN'T HERE TO ASK?? YOU'D FLUNK AND BE SENT BACK TO KINDERGARTEN, THAT'S WHAT!

SAYS YOU! I HEARD THAT SOMETIMES KIDS DON'T PAY ATTENTION BECAUSE THE CLASS GOES AT TOO SLOW OF A PACE FOR THEM. SOME OF US ARE TOO SMART FOR THE CLASS.

OH, RIGHT. YOU'RE TOO SMART.

BELIEVE IT, LADY. YOU KNOW HOW EINSTEIN GOT BAD GRADES AS A KID? WELL, MINNE ARE EVEN HORSE!

SO WHAT ARE WE SUPPOSED TO BE DOING?

WE'RE SUPPOSED TO BE RESEARCHING THE PLANET MERCURY.

SO WHAT HAVE WE FOUND OUT?

NOTHING! I'M NOT GOING TO DO THIS WHOLE THING MYSELF.

YOU'D PROBABLY GLOAT IT ALL UP IF YOU DID. LET'S GET STARTED.

YES! LET'S!

I'LL BE THE MANAGEMENT, AND YOU CAN BE THE LABOR. FIRST, GET SOME BOOKS.

DOES ANYONE WANT TO TRADE PARTNERS?
Differential Relations of Electrostatics

The divergence of the electrostatic field in vacuum is specified by the differential form of Gauss’ Law

\[ \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\varrho(\mathbf{r})}{\epsilon_0}. \]  

(1)

The curl of the electrostatic field is specified by the static form of Faraday’s Law

\[ \nabla \times \mathbf{E}(\mathbf{r}) = 0. \]  

(2)

By Helmholtz’ theorem, these two first-order vector differential relations completely determine the electrostatic field vector \( \mathbf{E}(\mathbf{r}) \) in any specified region of space given

1. the macroscopic charge density \( \varrho(\mathbf{r}) \) in that region, and
2. the appropriate boundary conditions on \( \mathbf{E}(\mathbf{r}) \).
As a consequence of (2), the electrostatic field may be expressed as

\[ \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) \]  

(3)

where \( V(\mathbf{r}) \) is the scalar potential.

- The negative sign in this relation indicates that the \( \mathbf{E} \)-field points toward a decrease in potential.
- Notice that \( V(\mathbf{r}) \) is not uniquely determined by Eq. (3) as one may add to it any quantity that has a zero gradient without changing \( \mathbf{E} \).
Poisson’s Equation for the Scalar Potential $V(\mathbf{r})$

Substitution of (3) into Gauss’ Law (1) then yields Poisson’s Equation

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$  \hspace{1cm} (4)

with integral solution [from Coulomb’s Law - see Eq. (26) in Topic 1]

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int\int\int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r',$$  \hspace{1cm} (5)

the integration extending over the entire region containing charge.

Notice that the principle of superposition applies to the electrostatic potential $V(\mathbf{r})$ just as it does to the electrostatic field vector $\mathbf{E}(\mathbf{r})$.

Problem 11: Show that the integral expression given in Eq. (5) for the electrostatic potential satisfies Poisson’s equation (4).
Siméon Denis Poisson (1781–1840)
In charge-free regions of space the electrostatic potential satisfies Laplace’s Equation

$$\nabla^2 V(r) = 0$$  \hspace{1cm} (6)

Because the second derivative measures the curvature of a function at a point, for example

$$\frac{\partial^2 \psi}{\partial x^2} = -2 \lim_{\Delta x \to 0} \frac{\psi(x, y, z) - \frac{1}{2} [\psi(x - \Delta x, y, z) + \psi(x + \Delta x, y, z)]}{(\Delta x)^2},$$

Laplace’s equation then states that there can never be an extremum in the electrostatic potential (either a maximum or minimum) in a charge-free region.
The general **boundary value problem (BVP)** of determining the electrostatic potential $V(r)$ corresponding to a given charge distribution $\varrho(r)$ in a particular region of space $\mathcal{R}$ then amounts to determining a solution to either Poisson’s equation (4) or Laplace’s equation (6) that will satisfy the given **boundary conditions** specified on the system of surfaces $\mathcal{S}$ enclosing the region $\mathcal{R}$.
Because $\mathbf{E} \cdot d\mathbf{\ell} = -\nabla V \cdot d\mathbf{\ell} = -dV$, then

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{\ell}$$

and the work expended in moving a test charge at a constant speed from a point $P_1$ to a point $P_2$ is independent of the path taken between those points (a conservative field).

Notice that the electric field intensity $\mathbf{E}(r)$ determines only the difference between the electrostatic potential at the two points.

For an open system, it is convenient to choose the potential at infinity to be zero. The electrostatic potential at a point $P$ is then given by

$$V(P) = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{\ell},$$

which is referred to as the absolute potential.
Work & Potential

The work $W$ required to bring a charge $q$ from a point at which the electrostatic potential is defined to be zero (ground) to a point at which the potential is $V$ is given by

$$W = qV \implies V = \frac{W}{q} \left( \frac{\text{joule}}{\text{coulomb}} \equiv \text{volt} \right) \quad (9)$$

Consider the electrostatic field produced by a single point charge $q$ at the origin. The absolute potential is then given by [from Eq. (8) with $d\vec{\ell} = \hat{r}dr$]

$$V(r) = -\frac{q}{4\pi \varepsilon_0} \int_\infty^r \frac{dr}{r^2} = \frac{q}{4\pi \varepsilon_0 r} \simeq (9 \times 10^9 \text{volt} \cdot \text{m/C}) \frac{q}{r} \quad (10)$$

and the sign of the potential $V$ is the same as that of the charge $q$. 

Equipotential Surfaces and Lines of Force

The set of points in space that are at a constant potential defines an **equipotential surface**. Because $V(\mathbf{r}) = constant$ on an equipotential, then

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = 0.$$  \hspace{1cm} (11)

Because $\mathbf{E} = -\nabla V$, the electric field is everywhere perpendicular to the equipotential surfaces. **Lines of force** are then defined such that they are everywhere perpendicular to the equipotential surfaces with $\mathbf{E}(\mathbf{r})$ tangent to these lines of force.
Convergence of the Scalar Potential

Consider the behavior of the scalar potential inside a differential element of space containing charge. Let the volume element \( d^3r \) be a spherical shell of thickness \( dr \) and radius \( r \) centered at the point \( P \). Then \( d^3r = 4\pi r^2 dr \) and the element of charge in this spherical shell produces a potential [see Eq. (5)]

\[
dV = \frac{\varrho}{4\pi \varepsilon_0} \frac{4\pi r^2 dr}{r} = \frac{\varrho}{\varepsilon_0} r dr,
\]

which remains finite as \( r \to 0 \).

The scalar potential for the electrostatic field then converges for sufficiently well-behaved charge density functions \( \varrho(r) \), the integral in Eq. (5) then being finite.
The mere continuity of the charge density \( \varrho(\mathbf{r}) \) in Poisson’s equation
\( \nabla^2 V(\mathbf{r}) = -\varrho(\mathbf{r})/\varepsilon_0 \) is not sufficient to ensure the existence of the second partial derivatives of the scalar potential \( V(\mathbf{r}) \).

**Definition (Hölder Condition):** A function \( f(Q) \) of the coordinates of a point \( Q \) is said to satisfy a Hölder condition at the point \( P \) iff there exists three positive constants \( A, \alpha, \) and \( r_0 \) such that

\[
|f(Q) - f(P)| \leq Ar^\alpha
\]  

(12)

for all points \( Q \) for which \( r \leq r_0 \), where \( r \) is the distance between the points \( P \) and \( Q \).
Uniqueness of Solution: Hölder Condition

If there is a region $\mathcal{R}$ in which $f(Q)$ satisfies a Hölder condition at every point $P$ for a fixed set of values for $A$, $\alpha$, and $r_0$, then the function $f(Q)$ is said to satisfy a uniform Hölder condition, or to satisfy a Hölder condition uniformly, in $\mathcal{R}$.

**Theorem (Hölder\(^1\))**: Let $V(r)$ be the potential of a source distribution with piecewise continuous density $\varrho(r)$ in a regular region $\mathcal{R}$. Then at any interior point $P_0 \in \mathcal{R}$ at which $\varrho$ satisfies a Hölder condition, the derivatives of $V(r)$ exist and satisfy Poisson’s equation.

However, this theorem leaves unanswered what boundary conditions on $V$ are necessary to obtain a unique and well-behaved solution of Poisson’s equation.

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\(^1\)Hölder, “Beiträge zur Potentialtheorie”, dissertation (Stuttgart, 1882).
Uniqueness of Solution: Boundary Conditions

It is expected that the following boundary conditions will lead to a unique solution of Poisson’s equation:

**Dirichlet Boundary Conditions:** The specification of the electrostatic potential $V(r)$ on a closed surface $S$, forming a Dirichlet Problem for the region $\mathcal{R}$ enclosed by $S$.

**Neumann Boundary Conditions:** The specification of the normal derivative $\partial V / \partial n$ of the electrostatic potential on a closed surface $S$, forming a Neumann Problem for the region $\mathcal{R}$ enclosed by $S$.

**Mixed Boundary Conditions:** The specification of Dirichlet boundary conditions on a portion $S_1$ of $S$ and Neumann boundary conditions on the remainder $S_2$ of $S$, where $S_1 \cap S_2 = \emptyset$, forming a Mixed Problem for the region $\mathcal{R}$ enclosed by $S = S_1 \cup S_2$. 
Uniqueness of Solution: Proof

Suppose that there are two solutions $V_1(r)$ and $V_2(r)$, each satisfying Poisson’s equation in the region $\mathcal{R}$, and suppose that both satisfy the same imposed boundary conditions on $S$. Then the function defined by

$$V(r) \equiv V_1(r) - V_2(r), \quad r \in \mathcal{R}$$

satisfies Laplace’s equation in the region $\mathcal{R}$; that is

$$\nabla^2 V(r) = 0, \quad r \in \mathcal{R},$$

and either $V(r) = 0$ on $S$ (Dirichlet bc’s), or $\partial V(r)/\partial n = 0$ on $S$ (Neumann bc’s), or $V(r) = 0$ on $S_1$ and $\partial V(r)/\partial n = 0$ on $S_2$ with $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$ (mixed bc’s).
Application of Green’s First Integral Identity [see Eq. (40) of Topic 2] to the difference function $V(r)$ then gives

$$\iiint_{\mathcal{R}} (V \nabla^2 V + \nabla V \cdot \nabla V) \, d^3r = \oint_{\mathcal{S}} V \frac{\partial V}{\partial n} \, d^2r,$$

where $\partial V / \partial n = \nabla V \cdot \hat{n}$. Because $\nabla^2 V = 0$ in $\mathcal{R}$ and either $V = 0$ or $\partial V / \partial n = 0$ on $\mathcal{S}$, then

$$\iiint_{\mathcal{R}} |\nabla V|^2 \, d^3r = 0,$$

which then implies that $\nabla V = 0$ everywhere in the region $\mathcal{R}$. Consequently, $V$ is a constant in $\mathcal{R}$. 
For Dirichlet boundary conditions, $V = 0$ on $S$ so that the constant must be zero and $V_1(\mathbf{r}) = V_2(\mathbf{r})$ in $\mathcal{R}$ and the solution is unique.

For Neumann boundary conditions, $\partial V / \partial n = 0$ on $S$ and, apart from an unimportant arbitrary additive constant, the solution is unique.

For mixed boundary conditions, $V = 0$ on part of $S$ so that the constant must be zero and $V_1(\mathbf{r}) = V_2(\mathbf{r})$ in $\mathcal{R}$ and the solution is unique.

However, a solution to Poisson’s equation when both $V$ and $\partial V / \partial n$ are specified arbitrarily on a closed boundary surface $S$, known as Cauchy boundary conditions, does not exist because there are separate unique solutions for the Dirichlet and Neumann boundary value problems and these will not, in general, be consistent.
WE HAVE TO GIVE OUR REPORT ON PLANET MERCURY TODAY. DID YOU DO YOUR HALF?

OF COURSE I DID. AND I’LL BET MY HALF MAKES YOUR HALF LOOK PATHETIC.

IT HAD BETTER BE GOOD... OR ELSE!

THE PLANET MERCURY AN EXHAUSTIVELY RESEARCHED REPORT BY CALVIN

"AND SO, THE PLANET MERCURY IS A HOT AND BARREN WORLD, THE CLOSEST TO OUR SUN."

AND TO TELL US ABOUT THE MYTHOLOGY OF MERCURY, HERE’S MY PARTNER, CALVIN.

THANK YOU, THANK YOU! HEY, WHAT A CROWD! YOU LOOK GREAT THIS MORNING... REALLY, I MEAN THAT! GO ON, GIVE YOURSELVES A HAND!

YOU KNOW, A FUNNY THING HAPPENED ON THE WAY TO THE LIBRARY YESTERDAY...

THIS ISN’T MY FAULT, MISS NORMANWOOD!

THE PLANET MERCURY WAS NAMED AFTER A ROMAN GOD WITH WINGED FEET.

MERCURY WAS THE GOD OF FLOWERS AND BOUQUETS, WHICH IS WHY TODAY HE IS A REGISTERED TRADEMARK OF FTD FLORISTS.

WHY THEY NAMED A PLANET AFTER THIS GUY, I CAN’T IMAGINE.

...UM... BACK TO YOU, SUSIE.
There are then several different approaches to solving a given electrostatic field problem:

- Coulomb’s law and linear superposition.
- Gauss’ law, taking advantage of the inherent geometric symmetry of the problem.
- Poisson’s or Laplace’s equation in a coordinate system that takes advantage of the inherent geometric symmetry of the problem.
Problem 12. Consider an equilateral triangle with sides of length $s$ & with equal point charges $+Q$ situated at each of the vertices, as illustrated. Choose the origin of coordinates $O = (0, 0)$ at the center of the triangle with $x, y$-coordinate axes in the plane of the triangle.

(a) Determine the electrostatic potential $V(x, y)$ in the $x, y$-plane.
(b) Determine the electrostatic field $\mathbf{E}(x, y) = -\nabla V(x, y)$.
(c) Determine the coordinate positions of all of the points in the finite $x, y$-plane at which the electrostatic field vanishes. Provide a physical explanation why $\mathbf{E} = \mathbf{0}$ at each such point.
Motivation (Part 4 - Post Mortem)

Boy, you should've seen the sparks fly when I gave my half of the report.

I've never seen Susie so mad. She accused me of not doing any research and claimed I made up the whole thing.

Did you? Heck, no. I just took a few creative liberties.

And they called your mom over a few creative liberties? Geez, you think Susie was mad...

Don't you hate it when your boogers freeze?

Here we are, overlooking Suicide Gulch, about to hurl ourselves down at breakneck speed in a sled that hardly steers!

Risking life and limb! Looking death straight in the eye!

'Why?' you ask! Why do we do it? Because we get paid, I hope. Because it's there!
The formal solution of either Poisson’s or Laplace’s equation in a finite region \( R \) of space with either Dirichlet or Neumann boundary conditions specified on the boundary surface \( S \) of \( R \) can be obtained through the use of Green’s second integral identity

\[
\iiint_{R} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3r = \oint_{S} \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \, d^2r,
\]

otherwise known as Green’s theorem, and the so-called Green’s functions.
This class of functions $G(r, r')$ is defined by the set of solutions of the equation
\begin{equation}
\nabla'^2 G(r, r') = -4\pi \delta(r - r'),
\end{equation}
where
\begin{equation}
G(r, r') = \frac{1}{|r - r'|} + F(r, r')
\end{equation}
with the function $F(r, r')$ satisfying Laplace’s equation within the interior of the region $\mathcal{R}$, so that
\begin{equation}
\nabla'^2 F(r, r') = 0; \quad r' \in \mathcal{R}.
\end{equation}

From a physical point of view, a Green’s function produces a response at an observation point $r$ due to a unit point source at $r'$. 

Let \( \phi(r') = V(r') \) be the scalar potential and let \( \psi(r') = G(r, r') \) be the Green’s function in Green’s theorem, which can then be solved for \( V(r) \) as

\[
V(r) = \frac{1}{4\pi \varepsilon_0} \iiint_{\mathcal{R}} \rho(r') G(r, r') d^3 r' + \frac{1}{4\pi} \oint_S \left[ G(r, r') \frac{\partial V(r')}{\partial n'} - V(r') \frac{\partial G(r, r')}{\partial n'} \right] d^2 r'.
\]

(17)

The freedom of choice afforded by the definition of the Green’s function given in Eqs. (14)–(16) then allows one to make the surface integral appearing in Eq. (17) depend only upon the type of boundary condition imposed.
For *Dirichlet boundary conditions* one demands that the Green’s function $G(r, r') = G_D(r, r')$ identically vanishes on the boundary surface $S$, so that

$$G_D(r, r') = 0, \quad r' \in S \quad (18)$$

and Eq. (17) becomes

$$V(r) = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{R}} \varrho(r') G_D(r, r') d^3r' - \frac{1}{4\pi} \iint_{S} V(r') \frac{\partial G_D(r, r')}{\partial n'} d^2r' \quad (19)$$

This Green’s function satisfies the symmetry property

$$G_D(r', r'') = G_D(r'', r') \quad (20)$$

which represents the physical interchangeability of the source and field points.
Neumann boundary conditions are a little more complicated due to Gauss’ theorem \[ \oint_S d\Omega = 4\pi \text{ if } P \in S \text{ and } \oint_S d\Omega = 0 \text{ if } P \notin S, \] where the solid angle is subtended at the point \( P \) which, when applied to Eq. (14), gives

\[
-4\pi = \iiint_R \nabla'^2 G(r, r') \, d^3 r' = \int_S \frac{\partial G(r, r')}{\partial n'} \, d^2 r'.
\]

As a consequence of this result, the simplest allowable boundary condition that can be imposed on the normal derivative of the Green’s function \( G(r, r') = G_N(r, r') \) with Neumann boundary conditions is

\[
\frac{\partial G_N(r, r')}{\partial n'} = -\frac{4\pi}{S}, \quad r' \in S, \tag{21}
\]

where \( S \) denotes the total surface area of the boundary surface \( S \).
With this substitution Eq. (17) becomes

\[
V(r) = \langle V \rangle_S + \frac{1}{4\pi \varepsilon_0} \iiint_{R} \varrho(r') G_N(r, r') d^3r' + \frac{1}{4\pi} \oint_{S} G_N(r') \frac{\partial V(r')}{\partial n'} d^2r'
\]

where \( \langle V \rangle_S = \frac{1}{S} \oint_{S} V(r') d^2r' \) denotes the average value of the potential over the entire boundary surface.

The typical Neumann boundary value problem is the so-called *exterior problem* in which the region \( R \) is bounded between two surfaces, one closed and at a finite distance from the origin of coordinates for the problem and the other at infinity. In this case the surface area \( S \) is infinite so that the boundary condition (21) on the Green’s function becomes homogeneous and the average value \( \langle V \rangle_S \) of the potential over \( S \) vanishes.
Because of the inhomogeneous nature of the Neumann boundary condition expressed in Eq. (21), the symmetry property under an interchange of source and field points is not automatic except in the homogeneous case of the exterior problem. Nevertheless, a symmetric Green’s function can always be constructed once $G_N(r', r'')$ has been determined by defining a modified Green’s function

$$\bar{G}_N(r', r'') \equiv G_N(r', r'') + \frac{1}{4\pi} \oint_S G_N(r'', r) \frac{\partial G_N(r', r)}{\partial n} d^2 r$$

which satisfies the Neumann boundary condition (21) by virtue of the fact that $\partial G_N(r', r)/\partial n$ is independent of its first argument. This modified Green’s function then satisfies the symmetry property

$$\bar{G}_N(r', r'') = \bar{G}_N(r'', r').$$
With these results a physical interpretation of the function $F(r, r')$ appearing in Eq. (15) may now be given. Because $F(r, r')$ is a solution of Laplace’s equation in the region $\mathcal{R}$ [see Eq. (16)], it then represents the electrostatic potential produced by a system of charges that are external to that region. For either Dirichlet or Neumann boundary conditions, the function $F(r, r')$ can then be regarded as the electrostatic potential due to an external distribution of charges that are chosen to satisfy either the homogeneous boundary conditions of zero potential or zero normal derivative of the potential on the boundary surface $S$, respectively, when combined with the potential produced by a unit point charge at the source point $r'$ appearing in the Green’s function (15). This interpretation then leads to the so-called method of images approach to solving specific types of boundary value problems in electrostatics that involve point charges in the presence of conducting bodies maintained at specified potential values.
For example, if the closed surface $S$ bounding the region $R$ is grounded so that $V(r') = 0$ for all $r' \in S$, then Eq. (19) becomes

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int\int\int_R \varrho(r') G_D(r, r') d^3 r'.$$  

This expression simply represents the principle of superposition as applied to both the distribution of point sources in $R$ with charge density $\varrho(r')$ and their corresponding image charges. Each elemental point source in $R$ contributes the differential element of potential $dV_1(r) = \frac{1}{4\pi\varepsilon_0} \frac{\varrho(r') d^3 r'}{|r - r'|}$ and each elemental image charge contributes the differential element of potential $dV_2(r) = \frac{1}{4\pi\varepsilon_0} \varrho(r') F(r, r') d^3 r'$, where $F(r, r') + \frac{1}{|r - r'|} = 0$ when $r' \in S$. The expression given in Eq. (25) then embodies the method of images.
If there are no charge sources in the region $R$ so that the potential satisfies Laplace’s equation $\nabla^2 V(r) = 0$ for all $r \in R$, then Eq. (19) becomes

$$V(r) = -\frac{1}{4\pi} \oint_S V(r') \nabla' G_D(r, r') \cdot \hat{n}' d^2 r',$$

(26)

where $\hat{n}'$ denotes the unit normal vector to $S$ directed out of the region $R$. The normal derivative $\frac{\partial}{\partial n'} G_D(r, r') = \nabla' G_D(r, r') \cdot \hat{n}'$ of the Green’s function at the surface represents the surface charge density

$$\rho_G(r') = \epsilon_0 \frac{\partial G_D(r, r')}{\partial n'} = \epsilon_0 \nabla' G_D(r, r') \cdot \hat{n}'$$

(27)

that would be induced on the boundary surface $S$ by a unit point charge at $r = r'$ if the boundary represented a grounded conductor. With this identification, Eq. (26) becomes

$$V(r) = -\frac{1}{4\pi \epsilon_0} \oint_S \rho_G(r') V(r') d^2 r'.$$

(28)
This last equation gives the solution to the potential problem corresponding to a specified potential \( V(r') \) on the boundary \( S \) of the region \( \mathcal{R} \) in terms of the surface integral of this potential multiplied by the charge density induced on the grounded boundary by a unit point charge placed at the field point \( r \in \mathcal{R} \). This expression also follows from Green’s reciprocation theorem (Provide a proof for next class.)

\[
\iint_{\mathcal{R}} \rho(r) \phi'(r) \, d^3r + \oint_{S} \rho_s(r) \phi'(r) \, d^2r = \iint_{\mathcal{R}} \rho'(r) \phi(r) \, d^3r + \oint_{S} \rho'_s(r) \phi(r) \, d^2r
\]

(29)

where \( \phi(r) \) is the electrostatic potential due to a volume charge density \( \rho(r) \) within \( \mathcal{R} \) and a surface charge density \( \rho_s(r) \) on the surface \( S \) bounding \( \mathcal{R} \), and \( \phi'(r) \) is the electrostatic potential due to another volume charge density \( \rho'(r) \) in \( \mathcal{R} \) and surface charge density \( \rho'_s(r) \) on \( S \).