

# Helmholtz' Theorem

EE 141 Lecture Notes  
Topic 3

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# Motivation



# Helmholtz' Theorem

Because

$$\nabla^2 \left( \frac{1}{R} \right) = -4\pi\delta(\mathbf{R}) \quad (1)$$

where  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$  with magnitude  $R = |\mathbf{R}|$  and where

$$\delta(\mathbf{R}) = \delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$$

is the three-dimensional [Dirac delta function](#), then any sufficiently well-behaved vector function  $\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z)$  can be represented as

$$\begin{aligned} \mathbf{F}(\mathbf{r}) &= \int_{\mathcal{V}} \mathbf{F}(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') d^3r' = -\frac{1}{4\pi} \int_{\mathcal{V}} \mathbf{F}(\mathbf{r}')\nabla^2 \left( \frac{1}{R} \right) d^3r' \\ &= -\frac{1}{4\pi} \nabla^2 \int_{\mathcal{V}} \frac{\mathbf{F}(\mathbf{r}')}{R} d^3r', \end{aligned} \quad (2)$$

the integration extending over any region  $\mathcal{V}$  that contains the point  $\mathbf{r}$ .

# Helmholtz' Theorem

With the identity  $\nabla \times \nabla \times = \nabla \nabla \cdot - \nabla^2$ , Eq. (2) may be written as

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi} \nabla \times \nabla \times \int_{\mathcal{V}} \frac{\mathbf{F}(\mathbf{r}')}{R} d^3 r' - \frac{1}{4\pi} \nabla \nabla \cdot \int_{\mathcal{V}} \frac{\mathbf{F}(\mathbf{r}')}{R} d^3 r'. \quad (3)$$

Consider first the divergence term appearing in this expression.

Because the vector differential operator  $\nabla = \hat{\mathbf{i}}_x \frac{\partial}{\partial x} + \hat{\mathbf{i}}_y \frac{\partial}{\partial y} + \hat{\mathbf{i}}_z \frac{\partial}{\partial z}$  does not operate on the primed coordinates  $\mathbf{r}' = \hat{\mathbf{i}}_x x' + \hat{\mathbf{i}}_y y' + \hat{\mathbf{i}}_z z'$ , then

$$\frac{1}{4\pi} \nabla \cdot \int_{\mathcal{V}} \frac{\mathbf{F}(\mathbf{r}')}{R} d^3 r' = \frac{1}{4\pi} \int_{\mathcal{V}} \mathbf{F}(\mathbf{r}') \cdot \nabla \left( \frac{1}{R} \right) d^3 r'. \quad (4)$$

# Helmholtz' Theorem

The integrand appearing in this expression may be expressed as

$$\begin{aligned}\mathbf{F}(\mathbf{r}') \cdot \nabla \left( \frac{1}{R} \right) &= -\mathbf{F}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{R} \right) \\ &= -\nabla' \cdot \left( \frac{\mathbf{F}(\mathbf{r}')}{R} \right) + \frac{1}{R} \nabla' \cdot \mathbf{F}(\mathbf{r}'),\end{aligned}\quad (5)$$

where the prime on  $\nabla'$  denotes differentiation with respect to the primed coordinates alone, viz.

$$\nabla' = \hat{\mathbf{i}}_x \frac{\partial}{\partial x'} + \hat{\mathbf{i}}_y \frac{\partial}{\partial y'} + \hat{\mathbf{i}}_z \frac{\partial}{\partial z'}$$

where  $\hat{\mathbf{i}}'_j = \hat{\mathbf{i}}_j$ ,  $j = x, y, z$ .

# Helmholtz' Theorem

Substitution of Eq. (5) into Eq. (4) and application of the divergence theorem to the first term then yields

$$\begin{aligned}\frac{1}{4\pi} \nabla \cdot \int_{\mathcal{V}} \frac{\mathbf{F}(\mathbf{r}')}{R} d^3 r' &= -\frac{1}{4\pi} \int_{\mathcal{V}} \nabla' \cdot \left( \frac{\mathbf{F}(\mathbf{r}')}{R} \right) d^3 r' \\ &\quad + \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{R} d^3 r' \\ &= -\frac{1}{4\pi} \oint_S \frac{1}{R} \mathbf{F}(\mathbf{r}') \cdot \hat{\mathbf{n}} d^2 r' \\ &\quad + \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{R} d^3 r' \\ &= \phi(\mathbf{r}),\end{aligned}\tag{6}$$

which is the desired form of the **scalar potential**  $\phi(\mathbf{r})$  for the vector field  $\mathbf{F}(\mathbf{r})$ . Here  $S$  is the surface that encloses the regular region  $\mathcal{V}$  containing the point  $\mathbf{r}$ .

# Helmholtz' Theorem

For the curl term appearing in Eq. (3) one has that

$$\begin{aligned}\frac{1}{4\pi} \nabla \times \int_{\mathcal{V}} \frac{\mathbf{F}(\mathbf{r}')}{R} d^3 r' &= -\frac{1}{4\pi} \int_{\mathcal{V}} \mathbf{F}(\mathbf{r}') \times \nabla \left( \frac{1}{R} \right) d^3 r' \\ &= \frac{1}{4\pi} \int_{\mathcal{V}} \mathbf{F}(\mathbf{r}') \times \nabla' \left( \frac{1}{R} \right) d^3 r'. \quad (7)\end{aligned}$$

Moreover, the integrand appearing in the final form of the integral in Eq. (7) may be expressed as

$$\mathbf{F}(\mathbf{r}') \times \nabla' \left( \frac{1}{R} \right) = \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{R} - \nabla' \times \left( \frac{\mathbf{F}(\mathbf{r}')}{R} \right), \quad (8)$$

so that

# Helmholtz' Theorem

$$\begin{aligned}\frac{1}{4\pi} \nabla \times \int_{\mathcal{V}} \frac{\mathbf{F}(\mathbf{r}')}{R} d^3 r' &= \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{R} d^3 r' \\ &\quad - \frac{1}{4\pi} \int_{\mathcal{V}} \nabla' \times \left( \frac{\mathbf{F}(\mathbf{r}')}{R} \right) d^3 r' \\ &= \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{R} d^3 r' \\ &\quad + \frac{1}{4\pi} \oint_S \frac{1}{R} \mathbf{F}(\mathbf{r}') \times \hat{\mathbf{n}} d^2 r' \\ &= \mathbf{a}(\mathbf{r}),\end{aligned}\tag{9}$$

which is the desired form of the **vector potential**.



# Helmholtz' Theorem

Equations (3), (6), and (9) then show that

$$\mathbf{F}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) + \nabla \times \mathbf{a}(\mathbf{r}) \quad (10)$$

where the scalar potential  $\phi(\mathbf{r})$  is given by Eq. (6) and the vector potential  $\mathbf{a}(\mathbf{r})$  by Eq. (9).

This expression may also be written as

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}_\ell(\mathbf{r}) + \mathbf{F}_t(\mathbf{r}) \quad (11)$$

known as the [Helmholtz decomposition](#).

# Helmholtz' Theorem

In the Helmholtz decomposition,

$$\begin{aligned}\mathbf{F}_\ell(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{1}{4\pi}\nabla\int_V\frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'+\frac{1}{4\pi}\nabla\oint_S\frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot\hat{\mathbf{n}}d^2r'\end{aligned}\quad (12)$$

is the **longitudinal or irrotational part of the vector field** (with  $\nabla\times\mathbf{F}_\ell(\mathbf{r}')=\mathbf{0}$ ), and

$$\begin{aligned}\mathbf{F}_t(\mathbf{r}) &= \nabla\times\mathbf{a}(\mathbf{r})=\frac{1}{4\pi}\nabla\times\nabla\times\int_V\frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \\ &= \frac{1}{4\pi}\nabla\times\int_V\frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'+\frac{1}{4\pi}\nabla\times\oint_S\frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\times\hat{\mathbf{n}}d^2r'\end{aligned}\quad (13)$$

is the **transverse or solenoidal part of the vector field** (with  $\nabla\cdot\mathbf{F}_t(\mathbf{r}')=0$ ).

# Helmholtz' Theorem

If the surface  $S$  recedes to infinity and if the vector field  $\mathbf{F}(\mathbf{r})$  is regular at infinity, then the surface integrals appearing in Eqs. (12)–(13) vanish and these two equations become

$$\begin{aligned}\mathbf{F}_\ell(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{1}{4\pi}\nabla\int_{\mathcal{V}}\frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r',\end{aligned}\tag{14}$$

$$\begin{aligned}\mathbf{F}_t(\mathbf{r}) &= \nabla\times\mathbf{a}(\mathbf{r}) \\ &= \frac{1}{4\pi}\nabla\times\int_{\mathcal{V}}\frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'.\end{aligned}\tag{15}$$

Taken together, the above results constitute what is known as **Helmholtz' theorem** or the **Fundamental Theorem of Vector Calculus**.

# Helmholtz' Theorem

## Theorem

**Helmholtz' Theorem.** *Let  $\mathbf{F}(\mathbf{r})$  be any continuous vector field with continuous first partial derivatives. Then  $\mathbf{F}(\mathbf{r})$  can be uniquely expressed in terms of the negative gradient of a scalar potential  $\phi(\mathbf{r})$  & the curl of a vector potential  $\mathbf{a}(\mathbf{r})$ , as embodied in Eqs. (10)–(11).*



Hermann Ludwig Ferdinand von Helmholtz (1821–1894)