Linking agent-based models and stochastic models of financial markets

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It is well-known that financial asset returns exhibit fat-tailed distributions and long-term memory. These empirical features are the main objectives of modeling efforts using (i) stochastic processes to quantitatively reproduce these features and (ii) agent-based simulations to understand the underlying microscopic interactions. After reviewing selected empirical and theoretical evidence documenting the behavior of traders, we construct an agent-based model to quantitatively demonstrate that “fat” tails in return distributions arise when traders share similar technical trading strategies and decisions. Extending our behavioral model to a stochastic model, we derive and explain a set of quantitative scaling relations of long-term memory from the empirical behavior of individual market participants. Our analysis provides a behavioral interpretation of the long-term memory of absolute and squared price returns: They are directly linked to the way investors evaluate their investments by applying technical strategies at different investment horizons, and this quantitative relationship is in agreement with empirical findings. Our approach provides a possible behavioral explanation for stochastic models for financial systems in general and provides a method to parameterize such models from market data rather than from statistical fitting.

Modeling price returns has become a central topic in the study of financial markets due to its key role in financial theory and its practical utility. Following models by Engle and Bollerslev (1, 2), many stochastic models have been proposed based on statistical studies of financial data to accurately reproduce price dynamics. In contrast to this stochastic approach, economists and physicists using the tools of statistical mechanics have adopted a bottom-up approach to simulate the same macroscopic regularity of price changes, with a focus on the behavior of individual market participants (3–10). Although the second so-called agent-based approach has provided a qualitative understanding of price mechanisms, it has not yet achieved sufficient quantitative accuracy to be widely accepted by practitioners.

Here, we combine the agent-based approach with the stochastic process approach and propose a model based on the empirically proven behavior of individual market participants that quantitatively reproduces fat-tailed return distributions and long-term memory properties (11–14).

Empirical and Theoretical Market Behaviors

We start by arguing that technical traders (usually agents seeking arbitrage opportunities and make their trading decisions based on price patterns) contribute much more to the dynamics of daily stock prices \( S_t \) (or log price \( \ln(S_t) \)) than fundamentalists (who attempt to determine the fundamental values of stocks). Although fundamentalists hold a majority of the stocks, they trade infrequently (see \textit{SI Appendix, Fig. S6}). In contrast, technical traders contribute most of the trading activities (15) by trading their minority holdings more frequently than fundamentalists.

Market surveys (16–18) also provide clear evidence of the prevalence of technical analysis. We consider here only technical traders, assuming that fundamentalists contribute only to market noise. Our study is of the empirical data recorded prior to 2006 and ignores the effect of high frequency trading (HFT) that has become significant only in the past 5 y. We propose a behavioral agent-based model that is in agreement with the following empirical evidence:

i. Random trading decisions made by agents on a daily basis. \( n_0 \) technical traders use different trading strategies, hence their decisions to buy, sell, or hold a position appear to be random. A trading decision is made daily because empirical studies report the lack of intraday trading persistence in empirical trading data (19). Market survey (16) also shows that fund managers put very little emphasis on intraday trades. We estimate the probability \( p \) of having daily trade empirically from trading volumes.

ii. Price returns. The price return \( r_t \equiv S_t - S_{t-1} \) is controlled by the imbalance \( d_t \) between the demand and the supply of stocks—the difference in the number of buy and sell trades each day. The excess in total demand or supply moves the price up or down, where the largest \( r_t \) occurs when all traders act in unison, when they all either buy or sell their stocks. We assume this relationship between price change \( r_t \) and \( d_t \) to be linear each day, as supported by empirical findings (20, 21).

iii. Centralized interaction mechanism of returns on technical strategies. For technical traders, an important input parameter in their strategies is past price movement (22, 23). Consequently, prices and orders reflect a main interaction mechanism between agents. In many agent-based models, the interaction strength between agents need to be adjusted with agent population size (5, 24, 25) or interaction structure (26) to sustain “fat” tails in return distributions. Here we propose a centralized interaction mechanism (price change) among agents so that the strength of interaction grows with agent population and is unaffected by interaction structure.

iv. Opinion convergence due to price changes. This is the unique mechanism that distinguishes our model from other models. It specifies the collective behavior of technical traders. Duffy et al. (27) found that agents learn from each other and tend to adopt the strategy that gives the most payoff. Given the price patterns at any point in time, a few most profitable technical strategies dominate the market because every technical trader

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wants to maximize his/her profit by using the most profitable strategy copied from each other (The most profitable strategy would become less profitable when most agents adopt it, and a new profitable strategy emerges from the new price trends; soon agents will flock to the new profitable strategy until it is no longer profitable. This is similar to the regime switching phenomenon in various agent-based models.). On the other hand, the individual strategies used by different technical traders differ in their parameterizations of the buy/sell time, amount of risk tolerated, or portfolio composition (15). So when the input signal—the previous price change $r_{t-1}$—is small, every agent acts independently. When the input signal is large, the agents act more in concert, irrespective of their differences in trading strategies. During the spreading panic of market crashes, for example, most agents sell their stocks (and market makers are likely to make losses in such circumstances). This supports the empirical finding that large price swings occur when the preponderance of trades have the same buy/sell decision indicated in the findings by Gabaix X. et al. (28).

### Behavioral Model and Results

Based on the items of evidence (i)–(iv) listed above we construct a three-step behavioral model:

- **Step 1.** Based on evidence (i), we assume $n_0$ agents, each of equal size 1. Each day, a trading decision $\psi_i(t)$ is made by each agent $i$,

  $$ \psi_i(t) \equiv \begin{cases} 1 & \text{with probability } p \Rightarrow \text{buy;} \\ -1 & \text{with probability } p \Rightarrow \text{sell;} \\ 0 & \text{with probability } 1 - 2p \Rightarrow \text{hold}. \end{cases} $$

  Trading volume is equal to the total number of trades in this case because every agent has the same trading size of 1. Hence daily trading volume $N_t$ is defined as

  $$ N_t \equiv \sum_{i=1}^{n_0} \psi_i(t) $$

  and $k$ is the sensitivity of price change with respect to $d_i$. We set $k$ to 1 because a choice for $k$ does not affect the statistical properties. Note that, according to Step 1, maximum (minimum) $d_i$ means that all agents are in collective mode when they behave the same, all of them either buy or sell stocks.

- **Step 2.** Based on evidence (ii), we define price change $r_t$ to be proportional to the aggregate demand $d_t$, i.e., the difference in the number of agents willing to buy and sell,

  $$ r_t \equiv kd_t = k \sum_{i=1}^{n_0} \psi_i(t). $$

- **Step 3.** Based on evidence (iii) and (iv), at day $t + 1$ each agent’s opinion is randomly distributed into each of the $c_{t+1}$ opinion groups

  $$ c_{t+1} = (n_0/|r_t|)^\omega, $$

  in which all agents comprising the same opinion group execute the same action (buy, sell, or hold) with the same probability $p$ as in Step 1. Because $1 \leq c_{t+1} \leq n_0$, Eq. 3 implies (a) when the previous return is maximum, $|r_t|_{\text{max}} = n_0$ (everyone buys/sells), there is only one trading opinion among the agents; (b) when the previous return is minimum $|r_t|_{\text{min}} = n_0^{(\omega - 1)/\omega}$ (see SI Appendix, Section 5), there are $n_0$ opinions and every agent acts independently of one another. We use $\omega = 1$ in most of our simulations because it produces results that are numerically close to empirical findings. The case of $\omega = 1$ corresponds to $|r_t|_{\text{min}} = 1$, and there is only one more buy/sell order than sell/buy order. The result with different $\omega$ is presented in the end of this section. When market noise is considered, $c_{t+1} \sim \mathcal{N}(n_0/|r_t|, \sigma_\epsilon^2)$, where $\mathcal{N}$ denotes the normal (Gaussian) distribution and $\sigma_\epsilon^2 = b \cdot n_0/|r_t|$ quantifies market noise due to external news events.

To obtain an empirical value for daily trading probability $p$, we choose 309 companies from the Standard and Poor’s 500 index traded over the 10-y period 1997–2006. Only 309 of the 500 total stocks were consistently listed during the entire 10-y period, and they are the ones we choose. We define the trading velocity $V$ of an agent as the total number of shares he/she trades in a year, divided by the number of shares, he/she owns on average over a year

$$ V = \frac{\text{Number of trades}}{\text{Number of shares}} $$

From the Compustat database, we divide the total number of shares traded on the market by the number of outstanding shares of each stock and obtain the average yearly trading velocity, $V$, for these stocks through the 10-y period. Thus we obtain,

$$ V = 1.64, $$

where the value of $V$ varies from 1.3 to 1.9 throughout the 10 y. Note that $V > 1$ means that, on average, each stock changes its owner more than once during a year. From the data documented by Yahoo! Finance, we assume that institutional owners are fundamentalists and that the rest of the traders are predominantly technical traders. The percentage of outstanding shares held by institutional owners is usually higher than 60%, and the average value is 83% (SI Appendix, Fig. S64).

Assuming institutional owners are fundamentalists with investment (trading) horizons longer than 1 y (they trade less frequently than once a year), we calculate the value of $p$ as follows. We let the average yearly trading velocity of fundamentalists be $V_f$, then the average yearly trading velocity of technical traders $V_t$ is

$$ V_t = (V - 0.83 V_f)/(1 - 0.83). $$

Because there are about 250 trading days in a year, we calculate the daily trading probability $p$

$$ p = V_t/(250 \cdot 2). $$

Because fundamentalists have an investment horizon longer than 1 y, $V_f < 1$ (see Eq. 4). We arbitrarily set $V_f$ to be 0.2, 0.4, 0.6, 0.8, and estimate the corresponding values of $p$ as 0.0174, 0.0154, 0.0134, 0.0115, respectively. For $(V_f, p) = (0.4, 0.0154)$, we find that approximately 80% of the trading volume is contributed by technical traders.

We compare the simulation results from our behavioral model (Steps 1, 2, and 3) with those obtained for the empirical set of stocks comprising the S&P 500 index. For the cumulative distribution (CDF) of both absolute returns and number of trades, Fig. 1 shows that the model reproduces the empirical data in both the central region and the tails. In particular, both model and empirical distributions have power-law tails (P$(|r_t| > x) \sim x^{-\xi}$, P$(n_i > x) \sim x^{-\xi}$) (13, 29–31) where the power-law exponents $\xi$ and $\bar{\xi}$ that we obtain for the empirical data by applying the Hill’s estimator are in agreement with the ones obtained for empirical data (32).

It is worth noting that with different values of $\omega$ in Eq. 3, the tail exponent $\xi$ for the absolute return distribution varies as
returns, defined as invariant with respect to different values of agent population size average, each opinion group has we can model price change as determined by Step 3. For simplicity, we use a normal distribution 10-y period 1997–2006. There are 8,000,000 data points in empirical data, and 1,000,000 sample points from simulation. (A) Cumulative distributions of daily returns, defined as \( r_t = \log(x_{t, \text{close}}) - \log(x_{t, \text{open}}) \), the difference between the daily opening and closing price of a stock on day \( t \). Thus we ignore overnight returns arising, e.g., due to news events. The price for each stock is normalized to zero mean and unit variance before aggregation into a single distribution. The simulation results agree with the shape of the empirical distribution. The Hill estimator on 1% of tail region gives \( \xi_{1997-2006} = 3.69 \pm 0.07 \), \( \xi_{\text{simulation}} = 2.06 \pm 0.08 \). (B) The cumulative distribution of the same set of data but analyzed on the daily number of trades. Because the number of trades increases each year, we normalize the data by each stock mean number of trades on a yearly basis before aggregating them into the distributions. The simulation again reproduces the empirical results. The Hill estimator applied to 2.5% of the tail region gives \( \xi_{1997-2006} = 4.02 \pm 0.07 \), \( \xi_{\text{simulation}} = 4.02 \pm 0.08 \). shown in Table 1. We find that \( \omega = 1 \) gives the tail exponent \( \xi_{\text{empirical}} \) that is closest to empirical finding (see SI Appendix, Section 5 for the implication of different values in \( \omega \)). Overall, we are able to generate power-law distributions in both returns and number of trades in agreement with empirical data. The fact that we are able to capture the trends in the two highly correlated quantities (33) implies a very plausible mechanism underlying our model. Furthermore, the tail exponent \( \xi_{\text{empirical}} \) is invariant with respect to different values of agent population size \( n_0 \) and fundamentalists’ investment horizon \( V_f \) (Table 2 and SI Appendix, Section 1). In particular, the insensitivity of the result to the number of agents \( n_0 \) distinguishes this model to most other agent-based models (34, 35).

**Extension to Stochastic Model**

Next we study the underlying stochastic process of our behavioral model (Step 1 to Step 3). The mechanism of opinion convergence under price changes is incorporated into a mathematical form for an analytical understanding. Step 3 in our behavioral model indicates that the daily price change is not deterministic, where its variance \( \sigma_t^2 = E[(r_t - E(r_t))^2] \) is related to total number of opinion groups \( c_t \), which is determined directly by previous return \( r_{t-1} \). On average, each opinion group has \( |r_{t-1}| \) agents, and we have

\[
\sigma_t^2 = c_t \cdot [p \cdot r_{t-1}^2 + p \cdot (-r_{t-1})^2 + (1 - 2p) \cdot 0] \\
\sigma_t^2 = 2p n_0 |r_{t-1}|. 
\]  

[8]

Because \( r_t \) presents the standard deviation of price change \( r_t \), we can model price change as \( r_t = \sigma_t n_0 \), where \( n_0 \) is a random variable with zero mean and unit variance, and its distribution is determined by Step 3. For simplicity, we use a normal distribution for \( \eta_t \), although we find that other distributions, such as t-distribution, give similar results in our analysis (SI Appendix, Section 4).

Again we note that the variance \( \sigma_t^2 \) in Eq. 8 is not constant but is time dependent, and time-dependent variances are commonly found in a variety of empirical outputs where phenomena are ranging from finance to physiology (2, 36). They are widely modeled with the Autoregressive Conditional Heteroskedasticity (ARCH) process (1). Here we provide a possible explanation for the ARCH effect found in financial data in terms of the behavior of technical traders—larger previous price change \( r_{t-1} \) brings traders’ opinions closer to each other, resulting in large subsequent price fluctuations. Theoretical analysis on Eq. 8 leads to the fat-tailed return distributions (SI Appendix, Section 2) ubiquitously observed in real data, and in our case a power-law tail. Previous works have generated ARCH effect from other mechanisms that is different from our model (37).

The time-dependent variance of Eq. 8 defined by the most recent price change \( r_{t-1} \) is therefore based on short memory in the previous \( r_t \). To this end, we further extend our behavioral model with two more items of empirical evidence:

v. Technical strategies are applied at different return intervals. In contrast to the simplistic realization in our behavioral model (Step 1 to Step 3) in which technical traders make their decisions only upon the most recent daily returns (see Eq. 8 and Step 3), market survey (16) indicates that technical strategies in practice are applied at different investment horizons ranging from 1 d to more than 1 y. This finding implies that agents calibrate their technical strategies based on returns of different time horizons, i.e., how stock preformed during the last day, week, month, up to year, or even longer. Most technical strategies are focused on short-term returns of a few days, and fewer are focused at yearly returns according to the survey (16).

vi. Increasing trading activity in volatile market conditions. Agents tend to trade more after large price movements. Different technical strategies set different thresholds on prices to trigger trading decisions (38), so large price fluctuations are likely to trigger more trades. Hence the probability of daily trade \( p \) is directly related to past returns. Because a large proportion of technical strategies is applied with short investment horizons, price changes from previous days have larger impact on \( p \) than price changes from past year.

**Table 1 Tail exponents of absolute return distribution with different \( \omega \)**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.4</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_{\text{empirical}} )</td>
<td>2.3</td>
<td>2.5</td>
<td>2.7</td>
<td>3.0</td>
<td>3.3</td>
<td>3.8</td>
<td>4.3</td>
<td>4.4</td>
<td>5.9</td>
<td>7.8</td>
<td>9.4</td>
</tr>
</tbody>
</table>
The survey on US market (16) allows us to estimate the relative proportion of technical analysis applied at different investment horizons. Precisely, agents having investment horizon $i$ days are affected by price change in the past $i$ days, and the relative portion of such traders is characterized by $\alpha_i$. Fig. 2 illustrates this plot, and we find that the curve follows a power-law decay with exponent $d = 1.12$, i.e., $\alpha_i \propto t^{-d}$. As different agents look at returns over time scales of differing lengths, their trading opinions are affected by the past returns of history longer than one day, so is the convergence of their collective opinions. Agents having investment horizon $i$ days are affected by price change in the past $i$ days, and the relative portion of such traders are characterized by $\alpha_i$. Hence the convergence of opinions is not only affected by the previous day's return but also the price difference of longer durations. Therefore Eq. 3 is better written as

$$c_{i+1} = \frac{n_0}{\sum_{i=1}^{M} \alpha_i |\gamma_i - s_{i-1}|},$$

where $M$ is the maximum investment horizon for which technical analysis is applied, and $\alpha_i$ is the proportion of agents focusing on investment horizon $i$ and it decays with $i$ as $\alpha_i \propto t^{-d}$ as in Fig. 2. When $M = 1$, Eq. 9 is identical to Eq. 3, which is the scenario for homogeneous investment horizon of 1 d. $|\gamma_i - s_{i-1}|$ is used as a simplified information content for technical traders with investment horizon $i$. Because there can be at least one opinion group, this boundary condition requires $\sum_{i=1}^{M} \alpha_i = 1$. From Eq. 9, Eq. 8 is transformed into

$$\sigma_{i+1}^2 = 2\sigma_0 \sum_{i=1}^{M} \alpha_i |\gamma_i - s_{i-1}| = 2\sigma_0 t^\alpha \Sigma_i,$$

where $\alpha \equiv \sum_{i=1}^{M} t^{-d} = 1$ and $\Sigma_i \equiv \sum_{i=1}^{M} t^{-d} |\gamma_i - s_{i-1}|$.

Note that Eq. 10 can be also intuitively derived. Because by $\alpha_i$ we denote the proportion of agents observing the price change in the past $i$ days, the price change $|\gamma_i - s_{i-1}|$ contributes to the overall opinion convergence with a weight $\alpha_i$. Even without knowing the exact contribution of $|\gamma_i - s_{i-1}|$ in the overall opinion convergence, by taking its first order effect with weight $\alpha_i$, we would have the functional form of Eq. 10.

To take into account the effect of increasing trading activity when market is more volatile, we assume that traders place technical thresholds based on returns of different horizons, and that the proportion of traders with different horizons is determined by the same survey results reported in ref. (16). Similar to the construction of cluster formation in Eq. 9, we define the time-dependent probability of trading as $p_{t-1} \approx p_0 + \alpha' \Sigma_t$. By $p_0$ we denote the base trading activity (largely due to fundamentalists) and $\alpha' \Sigma_t$ is the additional trade due to threshold crossing by technical traders. Therefore we transform Eq. 10 to

$$\sigma_{t+1}^2 \approx 2p_{t-1} n_0 \alpha \Sigma_t = 2a n_0 (p_0 + \alpha' \Sigma_t) \Sigma_t = \frac{a n_0 p_0^2}{2\alpha^2} + 2\alpha' n_0 \left( \Sigma_t + \frac{p_0}{2\alpha^2} \right)^2.$$

The first term is very small compared to the second term because technical traders dominate trading activities. Hence, we can ignore the first constant term and focus on the second quadratic term. Therefore, the previous equation transforms to

$$\sigma_{t+1} \approx \sqrt{2a\alpha'} \left( \Sigma_t + \frac{p_0}{2\alpha^2} \right) = A + B \Sigma_t.$$  

Constant $A$ is a scaling parameter defining the average size of returns, and $B$ characterizes the relative portion of trades accomplished by technical traders (due to crossing the thresholds) vs. background trading activities (mostly by fundamentalists). Eq. 11 has similar functional form as $\sigma_{t+1}$ in the fractional integrated models.
processes (39). However, there are some differences in a way how past returns are used in $\sigma_{t+1}$—whereas in fractional integrated process $\sigma_{t+1}$ depends on past daily returns, in Eq. 11 we use absolute values of past aggregate returns. Precisely, the absolute values of past daily, weekly, and monthly returns, to mention a few. Parke (40) has demonstrated how heterogeneous error durations among traders could result in fractional integration, and here we explicitly and quantitatively provide a behavioral interpretation of the long memory in absolute returns. The decaying dependence of standard deviation $\sigma_{t+1}$ on past aggregate returns of different durations in Eq. 11 is a direct outcome of agents applying technical strategies at different investment horizons. We find this dependence to decay as a power law, and the power-law exponent $d$ is calculated from empirical observations (16).

**Analytical and Simulated Results**

In general, the long memory in returns can be demonstrated by analyzing auto-correlation functions (ACF). We find that the ACFs of both absolute and squared returns ($\rho_\ell$ and $\rho_\ell^2$) decay as power laws with $\ell$, i.e., $\rho_\ell(|r_\ell|, |r_{\ell-\ell}|) \propto \ell^{-\gamma}$ and $\rho_\ell^2(r_\ell, r_{\ell-\ell}) \propto \ell^{-\gamma_2}$, where we obtain the scaling relations (SI Appendix, Section 2) similar to phase transitions in statistical mechanics,

$$
\begin{align}
\gamma_1 &= 2d - 2; \\
2d - 2 &\leq \gamma_2 \leq 4d - 4; \\
\gamma_1 &\leq \gamma_2 \leq 2\gamma_1.
\end{align}
$$

[12]

Whereas Ding et al. have shown that correlations in $|r_\ell|$ decay more slowly than correlations in $r_\ell^2$ (41), we, instead, show quantitative scaling relations that have been derived and explained by the behavior of individual market participants (characterized by $d$). Empirical verification with the stock components of Dow Jones Indices in Fig. 3 confirms the validity of the scaling relation of Eq. 12.

The behavioral understanding of our stochastic process Eq. 11 allows us to perform simulations in which every parameter is based on empirical calibration rather than on conventional statistical estimation. In Eq. 10 we replace the upper limit of $M$ in the summation by 500 trading days, which corresponds to $\approx2$ years of investment horizon as implied by the survey (16). We set $d = 1.2$, in agreement with empirical finding (16). From the empirical trading volume fluctuations and unconditional variance of price changes, we estimate $A = 0.002$ and $B = 0.05$ (SI Appendix, Section 3). Therefore we obtain $\sigma_{t+1} \approx 0.002 + 0.05 \sum_{i=1}^{500} i^{-2} |s_i - s_{i-1}|$. In Fig. 4 we compare the simulation results with the results for S&P 500 index. We demonstrate that the ACF of absolute and squared returns—for both simulations and the S&P 500 index—fulfill our scaling relations of Eq. 12 with $\gamma_{1,\text{simulation}} = 0.40$, $\gamma_{2,\text{simulation}} = 0.53$, $\gamma_{1,\text{S&P500}} = 0.44$, and $\gamma_{2,\text{S&P500}} = 0.7$.

Eq. 11 reflects the long-term memory of empirical data accurately while still being a stationary process (SI Appendix, Section 3) in contrast to many other power-law decaying processes (2, 39). In addition, the behavioral picture of Eq. 11 explains the origin of long-term memory through heterogeneous investment horizons of different technical traders based on empirical evidence.

**Summary**

Starting from the empirical behavior of agents, we construct an agent-based model that quantitatively explains the fat tails and long-term memory phenomena of financial time series without suffering from finite-size effects (35). The agent-based model and the derived stochastic model differ in construction but share the same mechanism of opinion convergence among technical traders. Whereas the agent-based model singles out the dominant market mechanism, the stochastic model allows a detailed analytical study. Both approaches allow their parameter values to be retrieved from market data with clear behavioral interpretations, thus allowing an in-depth study of this highly complex system of financial market.

The universality of various empirical features implies a dominant mechanism underlying market dynamics. Here we propose that this mechanism is driven by the use of technical analysis by market participants. In particular, past price fluctuations can directly induce convergence or divergence of agents’ trading decisions, which in turn give rise to the “ARCH effect” in empirical findings. Additionally, the heterogeneity in agents’ investment horizons gives rise to long-term memory in volatility.

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Fig. 4. Comparison between empirical data and simulation. The empirical data (about 10,000 data points) are from the S&P 500 index daily closing prices in the 40-y period 1971–2010. The simulation are carried out for 50,000 data points to obtain good convergence. Black Monday, October 19, 1989, is removed from the analysis of the ACF. (A) Autocorrelation of simulation vs. S&P 500 index. The simulation results show behavior similar to the S&P 500 for both absolute returns and squared returns. The rate of decay is also similar numerically. (B) The CDF of absolute returns for simulations and the S&P 500. The simulation reproduces the distribution well.
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SUPPLEMENTARY INFORMATION

This is the supplementary information for ‘Linking Agent-Based Models and Stochastic Models of Financial Markets’.

1. Robustness checking of behavioral model

In order to see how the parameters affect our behavioral model, we vary the values of probability of trading $p$, agent population size $n_0$ and market noise level $b$ to study the robustness of the model. Value of $p$ is determined indirectly by $V_f$ as shown in the paper, by varying the value of $V_f$ between 0.2 to 0.8, typically smaller than 1.0 due to the nature of fundamentalists’ investment horizons longer than one year. $n_0$ is chosen to vary from 256 to 16,384 covering a wide range. Market noise $b$ is defined in step 3 of the paper as: $c_{t+1} \sim \text{Normal}(n_0/|r_t|, \sigma_c^2)$, where $\sigma_c^2 = b \cdot n_0/|r_t|$. Value of $b$ is taken to be from 0 to 3.0 – no noise to extremely high noise. Figure 1 shows the simulation results of our behavioral model with different values of $V_f$ and $n_0$ varying individually. Each curve is shifted vertically for clarity. In 1a and 1b, it is

![Figure 1. CDF plot for different values of each parameter. The curves are shifted vertically for clarity. a) different values of $V_f$ b) different $n_0$. Different values of fundamentalists’ trading frequency $V_f$ and agent population size $n_0$ give almost identical results, showing the robustness of our model in producing the result matching empirical distributions.]
Figure 2. CDF plot for different values of noise $b$. The curves are shifted upwards for clarity. When market noise is high with a lot of news impacting the traders’ decisions, $b$ is big, and the tail of CDF becomes larger.

It is evident that the curves are almost identical in shape, demonstrating a robustness of the result against the variation in these two parameters. Hill estimator gives tail exponents close to empirical values, and the result is shown in the manuscript. Figure 2 shows the influence of market noise $b$ on the return distributions. Bigger noise gives fatter tail, resulting in huge price swings as expected.

2. Mathematical analysis on the stochastic process

*Excess Kurtosis of Behavioral Model*

In our behavioral model, the conditional kurtosis of $r_t$ is $Kurt(r_t| r_{t-1})$. Since there are $n_0/|r_{t-1}|$ number of independent clusters with $|r_{t-1}|$ agents each, the excess kurtosis is equal to the kurtosis contributed by each cluster divided by the number of clusters. The kurtosis of each cluster is $1/(2p)$, therefore we have the conditional excess kurtosis $Kurt(r_t| r_{t-1}) = (\frac{1}{2p} - 3)\frac{|r_{t-1}|}{n_0}$, which varies between $(\frac{1}{2p} - 3)/n_0$ to $(\frac{1}{2p} - 3)$. In the limit $n_0 \to \infty$, we have $0 \leq Kurt(r_t| r_{t-1}) \leq (\frac{1}{2p} - 3) \approx 30$ when
p = 0.015. Hence the conditional kurtosis of the distribution depends directly on previous returns.

For simplicity, we assume \( r_t = \sigma_t \eta_t \), where \( \eta_t \sim N(0,1) \) is a standard normal variable. It is shown later that t-distributions give similar simulation result. A simple calculation would show that this construction gives an unconditional kurtosis of

\[
Kurt(r) = \frac{\langle r_t^4 \rangle}{\langle r_t^2 \rangle^2} = \frac{\langle r_{t+1}^4 \rangle}{\langle r_{t+1}^2 \rangle^2}
\]

\[
= \frac{\langle \sigma_t^4 \eta_t^4 \rangle}{\langle 2pm_0|r_t| \rangle^2}
\]

\[
= \frac{\langle (2pm_0|r_t|)^2 \eta_t^4 \rangle}{\langle 2pm_0|r_t| \rangle^2}
\]

\[
= \frac{\langle \sigma_t^2 \eta_t^6 \rangle}{\langle |r_t|^2 \rangle^2}
\]

\[
> 3\left( \frac{\sigma_t}{|r_t|} \right)^2 = \frac{3\pi}{2}
\]

> 3.

This demonstrates the excess kurtosis of our time series model is in line with empirical observations.

It is worth noting that the real conditional distribution of \( \eta_t \) in our model is not Gaussian - it has larger kurtosis than Gaussian, and the conditional kurtosis value is varying according to our earlier analysis. But this should not affect our conclusions that the unconditional kurtosis is greater than 3, yet we could not conclude if the value is finite.

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*Autocorrelations of Stochastic Model*
As absolute values are difficult to deal with, we approximate them to $|s_t - s_{t-i}| \approx (\sum_{j=1}^{i} |r_{i-j}|)/\sqrt{i}$. Autocorrelation of absolute returns at lag $l$ is given by

$$
\rho_l \sim \langle |r_t| - \langle |r_t| \rangle \rangle \langle |r_{t+l}| - \langle |r_{t+l}| \rangle \rangle \\
\sim \langle |\sigma_t - \langle \sigma_t \rangle| \rangle \langle \sigma_{t+l} - \langle \sigma_{t+l} \rangle \rangle \\
\sim \langle \Sigma_t - \langle \Sigma_t \rangle \rangle \langle \Sigma_{t+l} - \langle \Sigma_{t+l} \rangle \rangle \\
\sim \langle \sum_{i,j=1}^{\infty} i^{-d-j^{-d}} (|s_t - s_{t-i}| - \langle |s_t - s_{t-i}| \rangle)(|s_{t+l} - s_{t+l-j}| - \langle |s_{t+l} - s_{t+l-j}| \rangle) \rangle \\
\sim \sum_{i,j=1}^{\infty} i^{-d-0.5} j^{-d-0.5} \Psi_{ijl},
$$

where $\Psi_{ijl} = \langle \sum_{m=1}^{i} |r_{t-m}| \sum_{n=1}^{j} |r_{t+l-n}| - \langle \sum_{m=1}^{i} |r_{t-m}| \rangle \langle \sum_{n=1}^{j} |r_{t+l-n}| \rangle \rangle$. When $j < l$, $\Psi_{ijl} \approx 0$. When $l < j < i + l$, we have $\Psi_{ijl} \approx \langle \sum_{m=1}^{k} |r_{t-m}| - \langle |r_{t-m}| \rangle \rangle^2 \approx k\sigma^2 |r| \approx (j-l)\sigma^2 |r|$. When $j > i + l$, we have $\Psi_{ijl} \approx \langle \sum_{m=1}^{k} |r_{t-m}| - \langle |r_{t-m}| \rangle \rangle^2 \approx i\sigma^2 |r|$. Therefore we have

$$
\rho_l \sim \sum_{i,j=1}^{\infty} i^{-d-0.5} j^{-d-0.5} \Psi_{ijl} \\
= \sum_{i \geq k}^{\infty} i^{-d-0.5} (k + l)^{-d-0.5} k\sigma^2 |r| + \sum_{k \geq i}^{\infty} i^{-d-0.5} (k + l)^{-d-0.5} i\sigma^2 |r|.
$$

Replacing the summation with continuous integral, we obtain

$$
\rho_l \sim \int_0^{\infty} \int_0^{\infty} i^{-d-0.5} k(k + l)^{-d-0.5} dk \; di \\
= \frac{1}{d-0.5} \int_0^{\infty} \frac{d^{1.5-d}}{(k + l)^{d+0.5}} dk + \frac{1}{1.5 - d} \int_0^{\infty} \frac{k^{1.5-d}}{(k + l)^{d+0.5}} dk \\
\sim B(1.5 - d, 2d - 1) i^{2-2d} \\
\propto l^{2-2d}.
$$

Therefore we obtain $\rho_l \propto l^{\gamma_1}$, where $\gamma_1 = 2 - 2d$. For autocorrelation in squared returns, we have

$$
\rho'_l \sim \langle (r_t^2 - \langle r_t^2 \rangle)(r_{t+l}^2 - \langle r_{t+l}^2 \rangle) \rangle \\
\sim \langle (\sigma_t^2 - \langle \sigma_t^2 \rangle)(\sigma_{t+l}^2 - \langle \sigma_{t+l}^2 \rangle) \rangle.
$$

There are two extreme situations here:
1) If \( p_0 \ll 2 \alpha' \Sigma_t \), where \( \Sigma_t \equiv \sum_{i=1}^{M} i^{-d}|s_t - s_{t-i}| \), representing the fact that trading tendency due to past price fluctuations is much higher than background value of \( p_0 \), we obtain \( \sigma_{t+1} \approx B \Sigma_t \). Assuming that \( (s_t - s_{t-i}) \) and \( (s_{t-i} - s_{t-j}) \) are weakly correlated, we obtain \( \text{corr}(|s_t - s_{t-i}|,|s_t - s_{t-j}|) \approx 0 \). Thus
\[
\sigma_t^2 - \langle \sigma_t^2 \rangle \approx 2 \sum_{i \leq j} (ij)^{-d} \langle |s_t - s_{t-i}||s_t - s_{t-j}| - \langle |s_t - s_{t-i}||s_t - s_{t-j}| \rangle \rangle
\]
\[
\approx 2 \sum_{i=1}^{\infty} i^{-2d} \langle (s_t - s_{t-i})^2 - \langle (s_t - s_{t-j})^2 \rangle \rangle.
\]
With similar treatment to \( \rho_l \), we obtain
\[
\rho'_l \sim \langle \sum_{i,j=1}^{\infty} i^{-2d} j^{-2d} ((s_t - s_{t-i})^2 - \langle (s_t - s_{t-i})^2 \rangle)((s_{t+l} - s_{t+l-j})^2 - \langle (s_{t+l} - s_{t+l-j})^2 \rangle) \rangle \sim l^{4-4d}.
\]
i.e. \( \rho'_l \sim \langle (r_t^2 - \langle r_t^2 \rangle)(r_{t+l}^2 - \langle r_{t+l}^2 \rangle) \rangle \sim l^{\gamma_2} \) where \( \gamma_2 = 4d - 4 \). For \( d = 1.12 \), we have \( \gamma_2 = 0.48 \).

2) In the case that \( p_0 \gg 2 \alpha' \Sigma_t \), which is approximately the case of our behavioral model, we again have the decay in squared returns as \( \rho'_l \sim l^{2-2d} \).

Therefore, the value of \( \gamma_2 \) fluctuates between \( \gamma_1 \) and \( 2\gamma_1 \) depending on the relative size of trades triggered by crossing thresholds placed by technical traders.

Overall, we obtained the scaling relations
\[
\gamma_1 = 2d - 2
\]
\[
2d - 2 \leq \gamma_2 \leq 4d - 4
\]
\[
\gamma_1 \leq \gamma_2 \leq 2\gamma_1.
\]

3. Determination of parameters in the stochastic process

The ratio of background trading volume \( p_0 \) against trading volume due to price impact \( \alpha \Sigma_t \) is approximated from empirical daily trading volume data of a stock. Since the market landscape has been changing during the last half a century due to computerization in late 90s and the adoption of high-frequency trading in the recent decade, there have been structural changes in trading volumes. Hence we only focus on the 10 year period 1997-2006 during which high frequency trading has not been massively adopted and program trading has already become a norm.
For stock AA, daily trading volume $v_i$ is used from 1997 to 2006 with $i \in [1, 2516]$. Since trading volume has increased through the ten-year period, we fit the data of $v_i$ v.s. $i$ with a linear function $y_i$, and form a new detrended time series of $v_i' = v_i/y_i$.

From $v_i'$, the average value of the smallest 1% of the data is assumed to be representing $p_0$, as we know that $p_t \geq p_0$. The average of the whole time series is assumed to be $\langle p_t \rangle$.

From our calculation, $\langle p_t \rangle \approx 3p_0$, which means $\langle p_0 + \alpha'\Sigma_t \rangle \approx 3p_0$. Hence we obtain the relation $p_0 \approx 0.5\langle \alpha'\Sigma_t \rangle$. This indicates trading volume due to price impact is twice of background trading volume $p_0$. This implies the relation

$$A = \frac{1}{4}B\langle \Sigma_t \rangle = \frac{1}{4}B \sum_{i=1}^{500} i^{-d} \langle |s_t - s_{t-i}| \rangle.$$ 

Therefore we have

$$\langle \sigma_t \rangle = A + B \sum_{i=1}^{500} i^{-d} \langle |s_t - s_{t-i}| \rangle$$

$$= \frac{5}{4}B \sum_{i=1}^{500} i^{-d} \langle |s_t - s_{t-i}| \rangle$$

$$\approx \frac{5}{4}B \sum_{i=1}^{500} i^{0.5-d} \langle |r_t| \rangle$$

$$= \frac{5}{4}B \sum_{i=1}^{500} i^{0.5-d} \langle \sigma_t \rangle \sqrt{2/\pi}$$

$$= \frac{5}{4} \sqrt{2/\pi} \sum_{i=1}^{500} i^{0.5-d} B \langle \sigma_t \rangle.$$ 

When $d = 1.2$, we have $B \approx 0.05$. The empirical value of $\langle \sigma_t \rangle \approx 0.01$ for S&P500, hence we have $A = \frac{1}{5}\langle \sigma_t \rangle = 0.002$. Analysis on other major stocks gives similar results. The estimation of $A$ and $B$ implies that the time series has finite unconditional variance.

4. Using t-distribution for $\eta_t$ instead of normal distribution

In our stochastic process, $r_t = \sigma_t \eta_t$, $\eta_t$ is assumed to follow standard normal distribution. This does not reflect the excess kurtosis of the conditional distribution discussed above. Hence we use t-distribution of different degrees of freedom to perform simulation. The degree of freedom is taken to be 6, 8, 10 so that the excess
kurtosis is 3, 1.5, 1. It can be seen from Figure 3 to 5 that the simulation results are still in agreement with empirical finding with different values of degree of freedom. The real underlying dynamics has a time varying distribution with excess kurtosis changing between 0 and 30 shown earlier. Since we demonstrate that fixed distributions with different excess kurtosis values give the same outcome, we can safely say the real dynamic process would exhibit the same scaling relations as well.

**Figure 3.** Plots of simulation results using $\eta_t$ following t-distribution with degree of freedom 6. Left) ACF plot of absolute and squared returns for simulation v.s. S&P500 index. Simulation gives $\gamma_1 = 0.38, \gamma_2 = 0.51$, which are in agreement with empirical finding. Right) CDF of absolute returns.
Figure 4. Plots of simulation results using $\eta_t$ following t-distribution with degree of freedom 8. Left) ACF plot of absolute and squared returns for simulation v.s. S&P500 index. Simulation gives $\gamma_1 = 0.31$, $\gamma_2 = 0.49$, which are in agreement with empirical finding. Right) CDF of absolute returns.

Figure 5. Plots of simulation results using $\eta_t$ following t-distribution with degree of freedom 10. Left) ACF plot of absolute and squared returns for simulation v.s. S&P500 index. Simulation gives $\gamma_1 = 0.38$, $\gamma_2 = 0.55$, which are in agreement with empirical finding. Right) CDF of absolute returns.

5. Analysis on different $\omega$ values in the equation of opinion cluster formation $c_{t+1} = \left(\frac{n_0}{|r_t|}\right)^\omega$

In Eq. (3) of the paper, we define the total number of opinion clusters at time $t+1$ as $c_{t+1}$, and it is directly determined by the previous price changes as $c_{t+1} =$
\((n_0/|r_t|)^\omega\). Since there is at least one opinion among the agents, and that happens when every agent does the same buy/sell action, we have the boundary condition that

\[ c_{t+1,\min} = 1 \text{ when } |r_t|_{\max} = n_0, \]

which fulfills \(c_{t+1} = (n_0/|r_t|)^\omega\) for any value of \(\omega\).

Since there are \(n_0\) agents in the market, there are at most \(n_0\) opinion groups in the market. According to Eq. (3), this occurs when

\[
c_{t+1,\max} = n_0 \\
(n_0/|r_t|_{\min})^\omega = n_0 \\
|r_t|_{\min} = n_0^{(\omega-1)/\omega}.
\]

This means whenever \(|r_t| = |r_t|_{\min} = n_0^{(\omega-1)/\omega}\), the market would have the maximum of \(n_0\) opinions and every agent acts independently.

When \(\omega = 1.0\), we have \(|r_t|_{\min} = 1\), and it means when there is exactly one more buy/sell trade than sell/buy trade in the market in the previous day, the market will have totally diverse opinions.

When \(\omega = 0.5\), we have \(|r_t|_{\min} = n_0^{-1}\). We have \(c_{t+1} = \sqrt{n_0} \ll c_{t+1,\max} = n_0\) (for large \(n_0\)) when \(|r_t| = 1\), meaning the market has some convergence in opinions when there is exactly one more buy/sell trade than sell/buy trade in the previous day. Hence the agents are more likely to converge in their opinions than the case for \(\omega = 1\), and return distribution has a fatter tail.

When \(\omega = 2.0\), we have \(|r_t|_{\min} = \sqrt{n_0}\). This means the market would have totally diverse opinions whenever \(|r_t|\) is smaller than \(\sqrt{n_0}\). Hence the agents are less likely to converge in their opinions than the case for \(\omega = 1\), and return distribution has a thinner tail.

According to our simulation result, \(\omega = 1\) gives the absolute return distribution closest to empirical findings.
Figure 6. a. Percentage of outstanding shares held by institutional investors of the 309 companies. Data is from Yahoo! Finance. Majority of the shares are held by institutional owners as the percentage is mostly larger than 70%. b. Estimated results on the average percentages of shares being held and traded by the two different types of traders – fundamentalists and technical traders. Data used are from Yahoo! Finance and COMPUSTAT. Left: Average percentage of outstanding shares held by institutional owners and technical traders. The majority stake is held by institutional owners who invest for long-term profits. Right: approximately 80% of trading volume is contributed by technical traders despite a minority holding by them. This number is calculated assuming fundamentalists have an investment horizon of 2.5 years. It is evident that although fundamentalists are holding the majority of the stocks, they only contribute a minority of the trading activity.