1. Show that a point $x$ for the map $f(x) = 3x \pmod{1}$ is eventually periodic if and only if it is a rational number. **Hint:** Let $x_0 = 0.a_1a_2\cdots a_Nb_1b_2\cdots b_p$ be a rational number written in base-3 (i.e., $x_0 = a_1\frac{1}{3} + a_2\frac{1}{9} + \cdots$). What happens to this IC under repeated iteration of the map? What if we could find no such $N$, i.e. if $x_0$ were irrational?

2. Construct the periodic table for the map $f(x) = 3x \pmod{1}$, up to period 10, using the same set of columns as in HW1. Careful... does the map $f$ have two fixed points or three? Note, $f^2(x) = 9x \pmod{1}$.

3. Consider the $3x$ (mod 1) map of the unit interval $[0,1]$. Define the distance between a pair of points $x, y$ to be either $|x - y|$ or $1 - |x - y|$, whichever is smaller. Measure with respect to the ‘circle metric’, in the sense of Figure 1.11, corresponding to the distance between two points on the circle.

(a) Show that the distance between any pair of points that lie within $1/6$ of one another is tripled by the map.

(b) Find a pair of points whose distance is not tripled by the map.

(c) Prove sensitive dependence for $x_0 = 0$ for this map, showing that $d$ can be taken to be any positive number less than $1/2$ in Definition 1.10, and $k$ can be chosen to be the smallest integer greater than $\ln(\frac{d}{|x_0-x|})/\ln(3)$. 

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4. Find the left and right endpoints of the subinterval LLR for the logistic map 
\( G(x) = 4x(1 - x) \).

5. Modify the matlab script logistic_period.m on the course website (or write one from
scratch in a programming language of your choosing) to compute the longest periodic
orbit you can find for the function \( g_a(x) = ax(1 - x) \). In other words, we know from
class that for particular values of the parameter \( a \), this map has a period-\( K \) orbit for
\( K = 2^N \), for any \( N \) you choose. You should provide me with 3 items: your code, \( a \), and
\( N \). Three bonus points will be awarded to the student finding the highest period orbit.
The course record is period \( 2^{25} \) using quadruple precision in C++. Eternal glory and
congratulatory tweets will be awarded if you’re able to beat \( 2^{25} \).

**Hint:** There are many possible approaches/answers for problem 5. If you are having
trouble, remember that a period-4 orbit will eventually (after transients) repeat every
fourth iterate. I have taken advantage of this fact in the example code. Please find a
value of the parameter \( a \) that results in an orbit of period 32 or higher. Note that if you
have taken Math 237, then I am expecting greatness out of you on this question, and
period 32 will not cut it.

**Matlab:** Here is a link to the code:
http://www.uvm.edu/~cdanfort/courses/266/matlab/logistic_period.m

If you’ve never used Matlab before, I suggest you watch this video:

I also suggest you ask another student who does know Matlab to help you get started. If
you can’t find such a person, you will need to log into one of the CEMS computers and
download the file linked above to a folder on your workspace. Keep the name
logistic_period.m as Matlab doesn’t like names with spaces ‘’ or dashes ‘-‘ which it
interprets as a minus sign.

Then double click on the Matlab icon on the computer, and use the url lookalike
navigation bar at the top of the Matlab window to navigate to the folder in which
you’ve saved logistic_period.m. Once you’ve done this, type
logistic_period
and press return, and in a few seconds you should the output:

This orbit repeats every 8 iterates.

Your task is to modify the parameter \( a \) to find higher period orbits. Do this modification
by changing ‘a’ in the code, saving the file, and running it again. The code is not clever
enough to function properly for really large periods without additional modification, e.g.
to the total number of iterates \( itotal \), the tolerance \( tol \) within which we assume we
have converged to a sink, and the largest periodic orbit (that repeats at a power of 2)
we can look for, \( 2^S \) where \( S = 10 \) by default in the code. Note that if you have found a
period \( 2^{10} = 1024 \) sink, you’ll need to adjust \( S \) to find a longer sink.