Math 266: Chaos, Fractals, & Dynamical Systems—Assignment 1
University of Vermont, Spring 2017

Due: By start of lecture, 8:30am, Thursday, January 26, 2017.
Some useful reminders:
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Instructions: Unless otherwise noted, use your brain and pencil to solve problems before checking with Matlab. Graduate students (required) and those planning to go to graduate school in a mathematical science (encouraged) should turn in their solutions in \LaTeX; you will need to learn this language eventually. Check the course website for sample m-files.
Grading: All questions are worth 3 points unless marked otherwise (3 = perfect or nearly so, 2 = close but something missing, 1 = not close but a reasonable attempt, 0 = way off). Excellent solutions will be returned with the graded HW.
Disclosure: Please show all your working clearly and list the names of other students with whom you collaborated.

1. Let \( p \) be a fixed point of a nonlinear map \( f \). Given an \( \epsilon > 0 \), find a geometric condition on \( f \) under which all points \( x \) in \( N_\epsilon(p) \) are in the basin of \( p \). Use cobweb plot analysis to explain your reasoning. **Hint:** By geometric condition, I mean some constraint on \( f \) and/or \( f' \) in the neighborhood of \( p \). One example condition that you can improve upon:

\[
\forall x \in (p - \epsilon, p), f(x) > x \& f(x) < p
\]

\[
\forall x \in (p, p + \epsilon), f(x) < x \& f(x) > p
\]

In more words and less notation: provided \( f(x) \) remains between the lines \( y = x \) and \( y = p \) in the epsilon neighborhood of \( p \), all points in the neighborhood are in the basin of \( p \). This question is deeper than you may think at first, and showing your condition works for a single example is not proving that it works for all example functions \( f \). Your condition needs to work for all \( f \).

2. The map \( f(x) = 2x^2 - 5x \) on \( \mathbb{R} \) has fixed points at \( x = 0 \) and \( x = 3 \).
   (a) Find a period two orbit for \( f \) by solving \( f^2(x) = x \) for \( x \).
   (b) What is the stability of the orbit?
   **Hint:** For this problem and the next, you will need to factor a degree 4 polynomial.
This can be done by hand without any horrific formulas if you think about what you already know about the roots.
3. Let $G(x) = 4x(1 - x)$.
   (a) Find the fixed points and period two points of $G$ and demonstrate that they are sources using $G'$ and/or $(G^2)'$.
   (b) Continue the periodic table for $G$ begun in Table 1.3. In particular how many periodic orbits of (minimum) period $k$ does $G$ have, for each $k \leq 10$?
   **Hint**: The pattern is not simple. Your table should look something like this:

<table>
<thead>
<tr>
<th>Period $k$</th>
<th>Number of fixed points of $G^k$</th>
<th>Proper divisors of $k$</th>
<th>Number of fixed points due to periods $&lt;k$</th>
<th>Number of fixed points due to period $k$ only</th>
<th>Orbits of period $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
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<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
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<tr>
<td>3</td>
<td>8</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>2</td>
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<tr>
<td>4</td>
<td>16</td>
<td>1, 2</td>
<td>4</td>
<td>12</td>
<td>3</td>
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</tbody>
</table>

4. Let $l(x) = ax + b$, where $a$ and $b$ are constants. For which values of $a$ and $b$ does $l$ have
   (a) an attracting fixed point?
   (b) a repelling fixed point?
   (c) a neutral point?

5. Let $x_1 < \ldots < x_8$ be the eight fixed points of $G^3(x)$ where $G(x) = 4x(1 - x)$. Clearly $x_1 = 0$.
   (a) For which $i$ is $x_i = \frac{3}{4}$?
   (b) Group the remaining six points into two orbits of three points each.
   **Hint**: It may help to consult Figure 1.10(c). The most elegant solution uses the chain rule. You need not compute the actual values of the $x_i$. 

2