The University of Vermont

METHODS Of APPLIED MATHEMATICS

Comprehensive Examination

January 2010

NAME:

ATTEMPT Six QUESTIONS Only

Passing requires four questions completely solved
and two other questions with substantial progress

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1. Show that the solution to the nonhomogenous Fredholm equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$$

where $K(x,t)$ is a Hilbert-Schmidt kernel, $f(x)$ and $y(x)$ are square integrable functions, and λ is not an eigenvalue of the homogeneous Fredholm equation, has solution of the form

$$y(x) = f(x) - \lambda \int_a^b \Gamma(x,t\,;\lambda)f(t)dt.$$

Here, the resolvent kernel $\Gamma(x,t\,;\lambda)$ can be expressed in terms of eigenvalues λ_k and orthonormal eiqnfunctions $\phi_k(x)$ of the homogeneous Fredholm equation as

$$\Gamma(x,t\,;\lambda) \;=\; \sum_{k=1}^{\infty} \frac{\phi_k(x)\,\phi_k(t)}{\lambda - \lambda_k}, \qquad (a \le x \le b\;\;, a \le t \le b)$$

2a. Verify the following order relations:
(i) $\varepsilon^2 \ln\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0^+$. (ii) $\sin\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0^+$.

2b. Given that $f(x)$ is continuous and has the asymptotic representation $f(x) \sim \sum_{n=0}^{\infty} a_n\, x^{-n}$ as $x \to \infty$, show that

$$F(x) \;=\; \int_x^{\infty} \Big(f(t) - a_0 - \frac{a_1}{t}\Big)dt \;\sim\; \sum_{n=1}^{\infty} \frac{a_{n+1}}{nx^n}\;.$$

3. Obtain the leading-order solutions of period 2π of the equation
$\ddot{x} \,+\, \Omega^2 x - \varepsilon x^2 \;=\; \Gamma\,\cos t\,, \qquad \varepsilon > 0$ when
(A) Ω is far from resonance and not close to an integer;
(B) $\Omega \approx 1$ and Γ is small. Note: In (B), assume that $\Omega^2 = 1 + \varepsilon\,\beta$ and $\Gamma = \varepsilon\gamma$.

4. Find the WKB approximation to the equation

$$\varepsilon^2 y'' \;-\; (1+x)^2\, y \;=\; 0\,, \qquad x > 0$$

with boundary conditions $y(0) = 0$ and $\lim_{x\to\infty} y(x) = 0$.

5. Use the method of steepest descent to obtain the asymptotic expansion for the integral

$$f(x) = \int_0^3 \ln t \; e^{ixt} dt, \qquad x \to \infty.$$

6. Construct a *leading-order approximation* to the solution, which is uniformly valid on $0 \le x \le 1$ for the problem

$$\varepsilon y'' \; + \; 2y' \; + \; y \; = \; 0\,, \qquad y(0) = 0, \quad y(1) = 1$$

where ε is small and positive.

7. The modified Bessel function $I_n(x)$ has the integral representation

$$I_n(x) \;\; = \;\; \frac{1}{\pi} \int_0^{\pi} \exp(x \cos\theta) \cos n\theta \, d\theta.$$

Show that $\qquad I_n(x) \;\sim\; \dfrac{e^x}{(2\pi x)^{1/2}}, \qquad x \to \infty.$

8. Use the Fredholm Alternative Theorem (without proof) to find conditions under which the nonhomogenous integral equation has a solution

$$y(x) \;\; = \;\; f(x) \; + \; \lambda \int_0^{\pi} (\cos^2 x \cos 2t \;\; + \;\; \cos 3x \cos^3 t)\, y(t)\, dt$$

for the following cases; (A) $f(x) = \cos x$ and (B) $f(x) = x$. Here λ is an eigenvalue of the corresponding homogeneous integral equation. (Hint: the values of λ should be obtained first.)

9. Find the asymptotic representation for $f(z)$, $|z| \to \infty$ in the sector $0 < |argz| < \pi/2$, where

$$f(z) \;\; = \;\; \int_0^{\infty} \frac{e^{-zt}}{1+t^4}\, dt.$$

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