COMPLEX VARIABLES PH.D. QUALIFYING EXAM

September 27, 2008

There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let $\mathbb{D}$ denote the open disc of radius 1 centered at the origin.

1. Let $f(z) = z|z|^2$.

(a) Find all points in the complex plane where f satisfies the Cauchy-Riemann equations.

(b) Does f have a complex derivative at the points you found in (a)? (Justify briefly.)

Solution: (a): One finds $u = x^3 + xy^2$, $v = x^2y + y^3$. The Cauchy-Riemann Equations require $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. This gives $3x^2 + y^2 = x^2 + 3y^2$ and $2xy = -2xy$. The latter implies $x = 0$ or $y = 0$, and then the former yields $x = 0$ and $y = 0$. The only point satisfying C-R is $z = 0$.

(b): Since both partials are continuous at $z = 0$ and C-R are satisfied, f does have a complex derivative there, namely $f'(0) = 0$. This can also be obtained directly as the limit of a difference quotient.

2. Find all complex numbers z such that $\tan z = i - 1$.

Solution: We use $\tan(z) = \sin(z)/\cos(z) = -i(e^{2iz} - 1)/(e^{21z} + 1)$. Substituting $w = e^{2iz}$, we obtain an easily-solved linear equation for w, yielding $w = (1 + 2i)/5$. Then solving $e^{2iz} = (1 + 2i)/5$, gives

$$2iz = -\tfrac{1}{2}\log(5) + i(\arg((1 + 2i)/5) + 2\pi in, \qquad n \in \mathbb{Z}$$

so $z = i\log(5)/4 + \arg(1 + 2i)/2 + \pi n$, for any integer n.

3. Let $f(z) = \overline{z}$. In each of parts (a) and (b) compute $\int_\gamma f(z)\,dz$, where γ is the specified path whose initial point is -1 and terminal point is i.

(a) γ is the path along the coordinate axes: from -1 to 0 and then from 0 to i.

(b) γ is the quarter of the unit circle lying in the second quadrant, oriented clockwise.

(c) Could there be a function g that is analytic on some simply connected open set U containing both the paths in (a) and (b) such that $g = f$ on both paths? (Explain briefly.)

Solution: (a): Parameterizing the paths leads to $\int_{-1}^{0} t\ dt + \int_0^1 (-it)i\ dt = -1/2 + 1/2 = 0$

(b): Using $z = -e^{-it}$, $0 \le t \le \pi/2$ leads to $\int_0^{\pi/2} -e^{it} i e^{-it}\, dt = -i\pi/2$.

(c): No. Integrals of any analytic function g over paths contained in a simply connected domain that have a common initial point and common terminal point result in the same value, contrary to what happened in (a) and (b).

4. Find a Laurent series expansion valid in some bounded annulus centered at 0 that contains the point $z = 3$ for the following function (explain briefly how the inner and outer radii of the annulus are determined):

$$f(z) = \frac{z}{z^2 - 4} + \frac{12}{(z-4)^2}.$$

Solution: We can write $f(z) = \frac{1}{z}\frac{1}{1-(4/z^2))} + \frac{3/4}{(1-(z/4))^2}$. Using the geometric series and its derivative, we find

$$f(z) = \left(\frac{1}{z} + \frac{4}{z^3} + \frac{4^2}{z^5} + \cdots\right) + \left(\frac{3}{4} + \frac{3}{4}(2)\left(\frac{z}{4}\right) + \frac{3}{4}(3)\left(\frac{z}{4}\right)^2 + \cdots\right)$$

The radii are determined as the min and max radii such that the open annulus they determine contains $z = 3$ and excludes all (non-removable) singularities of $f(z)$. In this case the radii are 2 and 4.

5. Use the calculus of residues to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 9}\, dx.$$

[Draw your path of integration; but you may quote without proof any standard estimates for integrals along portions of your path, making sure to mention what growth conditions are required.]

Solution: Let $f(z) = ze^{iz}/(z^2+9)$ so that the specified integral is the imaginary part of $\int_{-\infty}^{\infty} f(z)\, dz$. This is a standard Fourier transform integral of the type $\int_{-\infty}^{\infty} g(z)e^{iaz}\, dz$, where $a = 1$ and $g(z) = z/(z^2+9)$. One only needs the growth condition

$$|g(z)| \le K/|z| \qquad \text{for some } K \in \mathbb{R}^+$$

for all sufficiently large $|z|$ in order to integrate f over a rectangular path whose lower horizontal base is $[-A, B]$ on the real axis. The integrals along the remaining three sides go to zero as their lengths tend to infinity (see, for example, Lang's *Complex Analysis*, Section VI.2). Then $\int_{-\infty}^{\infty} f(z)\, dz = 2\pi i$(sum of the residues of f in the UHP). The only poles of f are at $\pm 3i$ with only $3i$ in the UHP. The residue at this simple pole is $e^{-3}/2$. Thus the value of the integral is the imaginary part of $2\pi i e^{-3}/2$ (which is pure imaginary), i.e., the integral is π/e^3.

6. Suppose f is entire and there is a positive real number M and a polynomial p such that $|f(z)| \leq |p(z)|$ for all z with $|z| > M$. Prove that f is a polynomial.

Solution: If p is a polynomial of degree d, then there are positive real constants A and B such that $|p(z)| \leq A + B|z|^d$ for all z with $|z| > M$. By a standard Cauchy estimate, the nth derivative of f vanishes at 0 for all $n > d$. Thus the Taylor series for f expanded about the origin is a polynomial.

7. Let f and g be analytic on the closed unit disc $\overline{\mathbb{D}}$, and assume both f and g have no zeros in $\overline{\mathbb{D}}$. Prove that if $|f(z)| = |g(z)|$ for all z with $|z| = 1$, then $f(z) = kg(z)$ in $\mathbb{D}$ for some constant k of modulus 1.

Solution: It is immediate from the hypotheses that both f/g and g/f are analytic on $\overline{\mathbb{D}}$. By the Maximum Modulus Principle both $|f/g|$ and $|g/f|$ are bounded by 1 on $\mathbb{D}$, hence $|f/g| = |g/f| = 1$ on $\overline{\mathbb{D}}$. Since f/g then has a maximum modulus in the open connected set $\mathbb{D}$, f/g must be constant on $\mathbb{D}$, hence $f/g = k$ for some $k \in \mathbb{C}$ of modulus 1.

8. (a) Exhibit a function f such that at each positive integer n, f has a pole of order n, and f is analytic and nonzero at every other complex number. (Briefly justify your answer.)

(b) Let f be any function that satisfies the conditions of part (a). For each positive integer N find $\displaystyle\int_{C_N} \frac{f'(z)}{f(z)}\, dz$, where C_N is the circle of radius $N + \frac{1}{2}$ centered at the origin.

Solution: First let z_n be the nth term in the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$. Note that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^3} = \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus by standard results on Weierstrass products the function

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp(z/z_n + (z/z_n)^2/2)$$

is entire and has zeros only at the positive integers, with the zero at n of order n, for all positive integers n. Thus $f = 1/g$ satisfies the conditions of (a).

(b): For any meromorphic function f, the poles of f'/f occur exactly at the points z_0 that are zeros or poles of f; and f'/f then has a simple pole at z_0 with residue equal to the order of zero or pole of f at z_0. For f as in (a), the only poles of f'/f are therefore at each positive integer n with residue $-n$. Exactly the first N of these lie inside C_N. By the Residue Theorem

$$\int_{C_N} \frac{f'(z)}{f(z)}\, dz = 2\pi i \sum_{n=1}^{N} -n = -\pi i n(n+1).$$

9. (a) State Goursat's Theorem.

(b) Use Goursat's Theorem to prove that if f is continuous on $\mathbb{C}$ and analytic at every point not on the real axis, then f must be analytic everywhere.

Solution: (a): Look it up.

(b): By Goursat's Theorem it suffices to show that the integral of f around any rectangle, R, whose sides are parallel to the axes is 0. If R lies entirely in either the upper or the lower half-plane, this is true by Goursat since f is analytic in these regions. If the vertical sides of R cross the real axis, divide R into two rectangles, R_1, R_2, that share a side lying on the real axis: the integral around R is the same as the sum of the integrals around R_1 and R_2 (the integrals along the real axis sides cancel because they are in reverse directions).

Thus we are reduced considering only the case when R equals R_1 or R_2, i.e., R has a side on the real axis. By symmetry we may assume R lies in the closed upper half-plane. Let $\epsilon > 0$ be given; we show $|\int_R f\ dz| < \epsilon$. As before, divide R into two rectangles, R_1, R_2, sharing a common side: R_1 entirely in the upper half-plane, and R_2 whose horizontal sides are the intervals $[a, b]$ and $[a+\delta i, b+\delta i]$ for some small δ to be determined. As before, $\int_{R_1} f\ dz = 0$. We finish the proof by showing $|\int_{R_2} f\ dz| < \epsilon$ for all sufficiently small δ. This is straightforward because by uniform continuity of f on compact sets containing R_2 we can choose δ sufficiently small so that $|f(x) - f(x + \delta i)| < \epsilon/2(b - a)$ for all $x \in [a, b]$. And by choosing δ sufficiently small, we can arrange that the integral of f over the vertical sides (each of length δ) is $< \epsilon/2$. Since the integrals along the top and bottom sides are in reverse directions, these estimates show $\int_{R_2} |f|\ dz$ can be made less than any given ϵ.

10. Suppose f is entire and for some positive real number K

$$|\text{Re } f(z)| \geq |\text{Im } f(z)|, \qquad \text{for all } z \text{ with } |z| \geq K.$$

Prove that f is constant on $\mathbb{C}$.

Solution: Let $g(z) = f(1/z)$ so the singularity of f at infinity is the same as the isolated singularity of g at 0, and the hypotheses for f hold likewise for g in some small disc about 0. Note that if f has a removable singularity at infinity, then it is entire and bounded, hence constant by Liouville. One must therefore show f does not have a pole or essential singularity at infinity; equivalently, g does not have a pole or essential singularity at 0. By hypothesis and Casorati-Weierstrass g does not have an essential singularity at 0 since the values of g would get arbitrarily close to i in any neighborhood of 0, thereby violating the hypothesis. Suppose g has a pole of order $n \geq 1$ at 0. In this case, the Laurent expansion of g about 0 gives

$$g(z) = \frac{a}{z^n}(1 + h(z)) \qquad \text{for some } a \in \mathbb{C} - \{0\} \tag{10.1}$$

in some neighborhood of zero where h is analytic with $h(0) = 0$. For M a positive real number let z_M be any n^{th} root of $\frac{a}{Mi}$. Now choose $M > 4$ and large enough so that z_M is in the neighborhood of convergence of (10.1), and $|1/z_M| \geq K$, and $|h(z_M)| < 1/4$. Then $g(z_M) = Mi(1 + h(z_M))$ is seen to violate the condition that $|\text{Re } g(z_M)| \geq |\text{Im } g(z_M)|$, a contradiciton.

One may also solve this question by invoking the Open Mapping Theorem.