PoCS What's The Story?

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number
University of Vermont, Fall 2023
Assignment 05
"Bitter, butter, Beetlejuice?"

Due: Sunday, October 8, by 11:59 pm
https://pdodds.w3.uvm.edu/teaching/courses/2023-2024pocsverse/assignments/05/
Some useful reminders:
Deliverator: Prof. Peter Sheridan Dodds (contact through Teams)
Assistant Deliverator: Chris O'Neil (contact through Teams)
Office: The Ether
Office hours: See Teams calendar
Course website: https://pdodds.w3.uvm.edu/teaching/courses/2023-2024pocsverse
Overleaf: LaTeX templates and settings for all assignments are available at
https://www.overleaf.com/read/tsxfwwmwdgxj.

All parts are worth 3 points unless marked otherwise. Please show all your workingses clearly and list the names of others with whom you ~~conspired~~ collaborated.

For coding, we recommend you improve your skills with Python, R, and/or Julia. The (evil) Deliverator uses (evil) Matlab.

Graduate students are requested to use LaTeX (or related TeX variant). If you are new to LaTeX, please endeavor to submit at least n questions per assignment in LaTeX, where n is the assignment number.

Assignment submission:

Via Brightspace or other preferred death vortex.

Please submit your project's current draft in pdf format via Brightspace by the same time specified for this assignment. For teams, please list all team member names clearly at the start.

Project start up details.

- Use this Overleaf LaTeX template:

  https://github.com/petersheridandodds/universal-paper-template

1. Everyday random walks and the Central Limit Theorem:

Show that the observation that the number of discrete random walks of duration $t = 2n$ starting at $x_0 = 0$ and ending at displacement $x_{2n} = 2k$ where $k \in \{0, \pm 1, \pm 2, \ldots, \pm n\}$ is

$$N(0, 2k, 2n) = \binom{2n}{n+k} = \binom{2n}{n-k}$$

leads to a Gaussian distribution for large $t = 2n$:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Please note that $k \ll n$.

Stirling's sterling approximation will prove most helpful.

Hint: You should be able to reach this form:

$$\frac{\text{Some stuff not involving spotted quokkas}}{\text{Some other quokka-free stuff} \times (1 - k^2/n^2)^{n+1/2}(1 + k/n)^k(1 - k/n)^{-k}}.$$

Lots of sneakiness here. You'll want to examine the natural log of the piece shown above, and see how it behaves for large n.

You may very well need to use the Taylor expansion $\ln(1 + z) \simeq z$.

Exponentiate and carry on.

Tip: If at any point quokkas appear in your expression, you're in real trouble. Get some fresh air and start again.

2. From lectures, show that the number of distinct 1-d random walk that start at $x = i$ and end at $x = j$ after t time steps is

$$N(i, j, t) = \binom{t}{(t + j - i)/2}.$$

Assume that j is reachable from i after t time steps.

Hint—Counting random walks:
http://www.youtube.com/watch?v=daSIYz-0U3E

3. (3 + 3)

Discrete random walks:

In class, we argued that the number of random walks returning to the origin for the first time after $2n$ time steps is given by

$$N_{\text{first return}}(2n) = N_{\text{fr}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$$

where

$$N(i,j,t) = \binom{t}{(t+j-i)/2}.$$

Find the leading order term for $N_{\mathrm{fr}}(2n)$ as $n \to \infty$.

Two-step approach:

(a) Combine the terms to form a single fraction,

(b) and then again use Stirling's bonza approximation.

If you enjoy this sort of thing, you may like to explore the same problem for random walks in higher dimensions. Seek out George Pólya.

And we are connecting to much other good stuff in combinatorics; more to come in the solutions.