## Mechanisms for Generating Power-Law Size Distributions, Part 1

 Principles of Complex Systems | @pocsvox CSYS/MATH 300, Fall, 2017
## Prof. Peter Dodds | @peterdodds

Dept. of Mathematics \& Statistics | Vermont Complex Systems Center Vermont Advanced Computing Core | University of Vermont

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## Random Walks

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## Power-Law

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Special Guest Executive Producer: Pratchett


0 On Instagram at pratchett the_cat[

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## Outline

# Random Walks <br> The First Return Problem <br> Examples 

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## References




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of Reality

## Mechanisms:

A powerful story in the rise of complexity:


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## A powerful story in the rise of complexity:

structure arises out of randomness.

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## Mechanisms:

A powerful story in the rise of complexity:
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© Exhibit A: Random walks. [

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## Mechanisms:

A powerful story in the rise of complexity:
structure arises out of randomness.
Exhibit A: Random walks. ©

The essential random walk:

- One spatial dimension.

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## Mechanisms:

A powerful story in the rise of complexity:
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Exhibit A: Random walks. ©

## The essential random walk:

O One spatial dimension.
Time and space are discrete
Random walker (e.g., a drunk) starts at origin

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A powerful story in the rise of complexity:
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Exhibit A: Random walks. ©

The essential random walk:

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A powerful story in the rise of complexity:
structure arises out of randomness.
. Exhibit A: Random walks. ©

The essential random walk:

- One spatial dimension.

Time and space are discrete
Random walker (e.g., a drunk) starts at origin $x=0$.
Step at time $t$ is $\epsilon_{t}$ :

$$
\epsilon_{t}= \begin{cases}+1 & \text { with probability } 1 / 2 \\ -1 & \text { with probability } 1 / 2\end{cases}
$$

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## A few random random walks:

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## Random walks:

Displacement after $t$ steps:

$$
x_{t}=\sum_{i=1}^{t} \epsilon_{i}
$$

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## Random walks:

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Displacement after $t$ steps:

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## Expected displacement:

$$
\left\langle x_{t}\right\rangle=\left\langle\sum_{i=1}^{t} \epsilon_{i}\right\rangle
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At any time step, we 'expect' our drunkard to be back at the pub. Obviously fails for ald number of steps. But as time gøes on, the chance of our drunkard lurching back to the pub must diminish, right?


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Mechanisms, Pt. 1
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Mechanisms, Pt. 1
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## Random walks:

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$\qquad$

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## Variances sum: []*

$$
\operatorname{Var}\left(x_{t}\right)=\operatorname{Var}\left(\sum_{i=1}^{t} \epsilon_{i}\right)
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* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.


## Variances sum: []*

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& =\sum_{i=1}^{t} \operatorname{Var}\left(\epsilon_{i}\right)=\sum_{i=1}^{t} 1
\end{aligned}
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## So typical displacement from the origin scales as:

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\sigma=t^{1 / 2}
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## So typical displacement from the origin scales as:

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\sigma=t^{1 / 2}
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A non-trivial scaling law arises out of additive aggregation or accumulation.


## Stock Market randomness:

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Also known as the bean machine[ $]$, the quincunx (simulation) ${ }^{\text {Cl }}$, and the Galton box.


## Great moments in Televised Random Walks:

## Plinko! © ${ }^{\top}$ from the Price is Right.



## Random walk basics:

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## Counting random walks:

> Each secific random walk of length $t$ appears with a chance $1 / 2^{t}$ Well bemore interested in how many random walks end up at the same place. Define $N(i, j, t)$ as \# distinct walks that start at

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## Random walk basics:

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Counting random walks:
Each specific random walk of length \(t\) appears with a chance \(1 / 2^{t}\).
\(\square\) Define \(N(\tau, j, t)\) as \# distinct walks that start at \(x=t\) and end at \(x=i\) after \(t\) time steps Random walk must displace by steps.

\section*{Random walk basics:}

\section*{Counting random walks:}
* Each specific random walk of length \(t\) appears with a chance \(1 / 2^{t}\).
We'll be more interested in how many random walks end up at the same place.

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Each specific random walk of length \(t\) appears with a chance \(1 / 2^{t}\).
- We'll be more interested in how many random walks end up at the same place.
- Define \(N(i, j, t)\) as \# distinct walks that start at \(x=i\) and end at \(x=j\) after \(t\) time steps.

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Each specific random walk of length \(t\) appears with a chance \(1 / 2^{t}\).
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- Define \(N(i, j, t)\) as \# distinct walks that start at \(x=i\) and end at \(x=j\) after \(t\) time steps.
R Random walk must displace by \(+(j-i)\) after \(t\) steps.

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Counting random walks:
Each specific random walk of length \(t\) appears with a chance \(1 / 2^{t}\).
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R Random walk must displace by \(+(j-i)\) after \(t\) steps.
\& Insert question from assignment 3 ■
\[
N(i, j, t)=\binom{t}{(t+j-i) / 2}
\]

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How does \(P\left(x_{t}\right)\) behave for large \(t\) ?
Take time \(t=2 n\) to help ourselves:


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For large \(n\), the binomial deliciously approaches the Normal Distribution of Snoredom:


How does \(P\left(x_{t}\right)\) behave for large \(t\) ?

\section*{Take time \(t=2 n\) to help ourselves.}
\(x_{2 n} \in\{0, \pm 2, \pm 4, \ldots, \pm 2 n\}\)
\(x_{2 n}\) is even so set \(x_{2 n}=2 k\).

For large \(n\), the binomial deliciously approaches the Normal Distribution of Snoredom:

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Using our expression \(N(i, j, t)\) with \(i=0, j=2 k\),
and \(t=2 n\), we have
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\operatorname{Pr}\left(x_{2 n}=2 k\right) \propto\binom{2 n}{n+k}
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For large \(n\), the binomial deliciously approaches the Normal Distribution of Snoredom:

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\operatorname{Pr}\left(x_{2 n} \equiv 2 k\right) \propto\binom{2 n}{n+k}
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\section*{Universality [ is also not left-handed:}

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\section*{Random walks are even weirder than you might think...}
> \(\xi_{r, t}=\) the probability that by time step t, a random walk has crossed the origin \(r\) times. Think of a coin flin oame with ten thou sand tosses. If you are behind early on, what are the chances you will make a comeback?

> Themostlikelymumher oflead changes is..

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In fact: \(\xi_{0, t}>\xi_{1, t}>\xi_{2, t}>\cdots\)

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Even crazier:
The expected time between tied scores \(=\infty\)

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The expected time between tied scores \(=\infty\)
See Feller, Intro to Probability Theory, Volume \({ }^{[3]}\)

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\section*{Applied knot theory:}

"Designing tie knots by random walks" [ᄌ
Fink and Mao,
Nature, 398, 31-32, 1999. \({ }^{[4]}\)

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Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie. a, The two ways of beginning a knot, \(L_{\odot}\) and \(L_{\otimes}\). For knots beginning with \(L_{\odot}\), the tie must begin inside-out \(\mathbf{b}\), The four-in-hand, denoted by the sequence \(L_{8} R_{\odot} L_{\theta} C_{\odot} T\). \(\mathbf{c}, A\) knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk \(\hat{1} \hat{\mathrm{r}} \hat{\mathrm{l}} \hat{\mathrm{c}}\).

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\section*{Applied knot theory:}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline h & \(\gamma\) & r/h & \(K(h, \gamma)\) & s & \(b\) & Name & Sequence \\
\hline 3 & 1 & 0.33 & 1 & 0 & 0 & & \(L_{\odot} \mathrm{R}_{\otimes} \mathrm{C}_{\odot} \mathrm{T}\) \\
\hline 4 & 1 & 0.25 & 1 & -1 & 1 & Four-in-hand & \(L_{*} \mathrm{R}_{\odot} L_{\otimes} \mathrm{C}_{\odot} T\) \\
\hline 5 & 2 & 0.40 & 2 & -1 & 0 & Pratt knot & \(L_{\odot} C_{\otimes} R_{\odot} L_{\otimes} C_{\odot} T\) \\
\hline 6 & 2 & 0.33 & 4 & 0 & 0 & Half-Windsor & \(L_{8} R_{\odot} C_{\otimes} L_{\odot} R_{8} C_{\odot} T\) \\
\hline 7 & 2 & 0.29 & 6 & -1 & 1 & & \(L_{\odot} R_{8} L_{\odot} C_{8} R_{\odot} L_{8} C_{\odot} T\) \\
\hline 7 & 3 & 0.43 & 4 & 0 & 1 & & \(L_{\odot} C_{\otimes} R_{\odot} C_{\otimes} L_{\odot} R_{\otimes} C_{\odot} T\) \\
\hline 8 & 2 & 0.25 & 8 & 0 & 2 & & \(L_{8} R_{\odot} L_{8} C_{\odot} R_{8} L_{\odot} R_{8} C_{\odot} T\) \\
\hline 8 & 3 & 0.38 & 12 & -1. & 0 & Windsor & \(L_{\otimes} C_{\odot} R_{8} L_{\odot} C_{\otimes} R_{\odot} L_{\otimes} C_{\odot} T\) \\
\hline 9 & 3 & 0.33 & 24 & 0 & 0 & & \(L_{\odot} R_{\otimes} C_{\odot} L_{\otimes} R_{\odot} C_{8} L_{\odot} R_{\otimes} C_{\odot} T\) \\
\hline 9 & 4 & 0.44 & 8 & -1 & 2 & & \(\mathrm{L}_{\odot} \mathrm{C}_{\otimes} \mathrm{R}_{\odot} \mathrm{C}_{\otimes} \mathrm{L}_{\odot} \mathrm{C}_{\otimes} \mathrm{R}_{\odot} \mathrm{L}_{\otimes} \mathrm{C}_{\odot} T\) \\
\hline
\end{tabular}

Knots are characterized by half-winding number \(h\), centre number \(\gamma\), centre fraction \(\gamma / h\), knots per class \(K(h, \gamma)\), symmetry \(s\), balance \(b\), name and sequence.

\section*{噱 \(h=\) number of moves}
\& \(\gamma=\) number of center moves
\& \(K(h, \gamma)=\) \(2^{\gamma-1}\binom{h-\gamma-2}{\gamma-1}\)
s \(s=\sum_{i=1}^{h} x_{i}\) where \(x=-1\) for \(L\) and +1 for \(R\).
R \(b=\frac{1}{2} \sum_{i=2}^{h-1}\left|\omega_{i}+\omega_{i-1}\right|\)
where \(\omega= \pm 1\)
represents winding direction.

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\section*{Random walks \#crazytownbananapants}

The problem of first return:


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\section*{Random walks \#crazytownbananapants}

\section*{The problem of first return:}

What is the probability that a random walker in one dimension returns to the origin for the first time after \(t\) steps?

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The problem of first return:
What is the probability that a random walker in one dimension returns to the origin for the first time after \(t\) steps?
Will our drunkard always return to the origin?

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Will our drunkard always return to the origin?
What about higher dimensions?

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\section*{Reasons for caring:}

We will find a power-law size distribution with an

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\section*{Random walks \#crazytownbananapants}

\section*{The problem of first return:}

What is the probability that a random walker in one dimension returns to the origin for the first time after \(t\) steps?
Will our drunkard always return to the origin?
What about higher dimensions?
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\section*{Reasons for caring:}
1. We will find a power-law size distribution with an interesting exponent.


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\section*{Reasons for caring:}
1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

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\section*{For random walks in 1-d:}

to origin can only happen when \(t=2 n\).

Call \(P_{f r(2 n)}\) the probability of
Prohahility calculation = Counting problem
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Idea: Transform first return problem into an
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\section*{For random walks in 1－\(d\) ：}


A return to origin can only happen when \(t=2 n\) ．

\(\square\)

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For random walks in 1-d:


A return to origin can only happen when \(t=2 n\).
In example above, returns occur at \(t=8,10\), and 14.


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Can assume drunkard first lurches to \(x=1\).

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Can assume drunkard first lurches to \(x=1\).
Observe walk first returning at \(t=16\) stays at or above \(x=1\) for \(1 \leq t \leq 15\) (dashed red line).




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Random Walks
The First Return Problem

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The 2 accounts for drunkards that first lurch to \(x=-1\).


\section*{Counting first returns:}

\section*{Approach:}
Move to counting numbers of walks.
Return to probability at end Again Akf; \(;\) tis the \# of possible walks between \(x=i\) and \(x=j\) taking \(t\) steps.
Consider all paths starting at \(r=1\) and ending at\(x=1\) after \(t=2 n-2\) steps.Idea: If we can compute the number of walks thathit \(x=0\) at least once, then we can subtract thisfrom the total number to find the ones thatmaintain \(x \geq 1\)Call walks that drop below \(x=1\)We'll use a methiod of images to identify theseexcluded walks.


\section*{Counting first returns:}

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Move to counting numbers of walks.

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The First Return Problem -......................... Examples

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Examples of excluded walks:



\section*{Key observation for excluded walks:}

For any path starting at \(x=1\) that hits 0 , there is a unique matching path starting at \(x=-1\).

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Examples of excluded walks:



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For any path starting at \(x=1\) that hits 0 , there is a unique matching path starting at \(x=-1\).
8 Matching path first mirrors and then tracks after first reaching \(x=0\).
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\section*{Random Walks}

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So \(N_{\text {first return }}(2 n)=N(1,1,2 n-2)-N(-1,1,2 n-2)\)

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\section*{Probability of first return:}
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Insert question from assignment 3 [ 3 :
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## Probability of first return:

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## Insert question from assignment 3 [3:

Find

$$
N_{\mathrm{fr}}(2 n) \sim \frac{2^{2 n-3 / 2}}{\sqrt{2 \pi} n^{3 / 2}} .
$$

Normalized number of paths gives probability.

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## Probability of first return:

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Normalized number of paths gives probability. Total number of possible paths \(=2^{2 n}\).

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$$
P_{\mathrm{fr}}(2 n)=\frac{1}{2^{2 n}} N_{\mathrm{fr}}(2 n)
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We have $P(t) \alpha \cdot t^{-3 / 2}, \gamma=3 / 2$
Same scaling holds for continuous space/time walks. $p(f)$ is normalizable.

Recurence: Random walker always returns to origin
But meam variance, and all higher moments are infinite.

Even though walker must return, expect a long wait... Reneated gambling against an infinitely wealthy opponent must lead to ruin.

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Higher dimensions [J:

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Higher dimensions [ 3 :
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Assoclated genias:
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\＆Associated genius：George Pólya［ $چ$

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## Random walks

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## On finite spaces:

In any finite homogeneous space, a random walker will visit every site with equal probability Call this onrobability the Invariant Deinsity of a dynamical system

Non-trivial Invariant Densities arise in chaotic
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## Random walks

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On finite spaces:
In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
Non-trivial Invariant Densities arise in chaotic systems.

\section*{On networks:}

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\section*{Random walks}

On finite spaces:
In any finite homogeneous space, a random walker will visit every site with equal probability
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Non-trivial Invariant Densities arise in chaotic systems.

On networks:
On networks, a random walker visits each node with frequency \(\propto\) node degree \#groovy


\section*{Random walks}

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On networks:
On networks, a random walker visits each node with frequency \(\propto\) node degree
Equal probability still present: walkers traverse edges with equal frequency.

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\section*{Scheidegger Networks \({ }^{[9,2]}\)}

Random directed network on triangular lattice.
- Toy model of real networks.
'Flow' is southeast or southwest with equal probability.


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\section*{Scheidegger networks}

Creates basins with random walk boundaries.

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\section*{Scheidegger networks}

Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:
\[
\epsilon_{t}=\left\{\begin{array}{cl}
+1 & \text { with probability } 1 / 4 \\
0 & \text { with probability } 1 / 2 \\
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\end{array}\right.
\]

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Random walk with probabilistic pauses.
Basin ter
problem: Basinlength r-uncolinimblut instribution: networks, qeneralizeto

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Basin length \(\ell\) distribution: \(P(\ell) \propto \ell^{-3 / 2}\)

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Random walk with probabilistic pauses.
B Basin termination = first return random walk problem.
Basin length \(\ell\) distribution: \(P(\ell) \propto \ell^{-3 / 2}\)
For real river networks, generalize to \(P(\ell) \propto \ell^{-\gamma}\).

\section*{Connections between exponents：}

For a basin of length \(\ell\) ，width \(\propto \ell^{1 / 2}\)

\section*{Connections between exponents:}

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For a basin of length \(\ell\), width \(\propto \ell^{1 / 2}\)
Basin area \(a \propto \ell \cdot \ell^{1 / 2}=\ell^{3 / 2}\)

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\section*{Connections between exponents:}

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For a basin of length \(\ell\), width \(\propto \ell^{1 / 2}\)
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Invert: \(\ell \propto a^{2 / 3}\)

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\section*{Connections between exponents:}

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\(\mathrm{d} \ell \propto \mathrm{d}\left(a^{2 / 3}\right)=2 / 3 a^{-1 / 3} \mathrm{~d} a\)

\section*{Connections between exponents:}

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\& \(\mathrm{d} \ell \propto \mathrm{d}\left(a^{2 / 3}\right)=2 / 3 a^{-1 / 3} \mathrm{~d} a\)
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\(=a^{-\tau} \mathrm{d} a\)

\section*{Connections between exponents:}

\section*{Connections between exponents：}

\section*{Both basin area and length obey power law distributions}

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\section*{Connections between exponents:}

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Both basin area and length obey power law distributions \\ Observed for real river networks
}

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\section*{Connections between exponents:}
Both basin area and length obey power law distributions
Observed for real river networks
Reportedly: \(1.3<\tau<1.5\) and \(1.5<\gamma<2\)

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Generalize relationship between area and length:

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\& Hack's law \({ }^{[5]}\) :
\[
\ell \propto a^{h} .
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Smaller basins possibly \(h>1 / 2\) (see: allometry).

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Models exist with interesting values of \(h\).
Plan: Redo calc with \(\gamma, \tau\), and \(h\).

\section*{Connections between exponents:}

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\section*{Connections between exponents:}

Power-Law
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\section*{Given}
\[
\ell \propto a^{h}, P(a) \propto a^{-\tau}, \text { and } P(\ell) \propto \ell^{-\gamma}
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8
Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

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\section*{Connections between exponents:}

With more detailed description of network structure, \(\tau=1+h(\gamma-1)\) simplifies to:
\[
\tau=2-h
\]
and
\[
\gamma=1 / h
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Only one exponent is independent (take \(h\) ).


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Simplifies system description．


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Only one exponent is independent (take \(h\) ).
Simplifies system description.
Expect Scaling Relations where power laws are found.

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Only one exponent is independent (take \(h\) ).
Simplifies system description.
Expect Scaling Relations where power laws are found.
Reed only characterize Universality class with independent exponents.

\section*{Other First Returns or First Passage Times:}

PoCS | @poesvox

\section*{Failure:}

A very simple model of failure/death:
. \(x_{t}=\) entity's 'health' at time \(t\)
Start with \(x_{0}>0\).
Entity fails when \(x\) hits 0 .

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\section*{Other First Returns or First Passage Times:}

\section*{Failure:}

A very simple model of failure/death:
. \(x_{t}=\) entity's 'health' at time \(t\)
- Start with \(x_{0}>0\).

Entity fails when \(x\) hits 0 .

\section*{Streams}

Dispersion of suspended sediments in streams.
Long times for clearing.

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\section*{More than randomness}

\section*{Can generalize to Fractional Random Walks \({ }^{[7, ~ 8, ~ 6] ~}\)}
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\section*{More than randomness}

\title{
( Can generalize to Fractional Random Walks \({ }^{[7, ~ 8, ~ 6] ~}\) Levy flights, Fractional Brownian Motion
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8 See Montroll and Shlesinger for example: \({ }^{[6]}\) "On \(1 / f\) noise and other distributions with long tails."
Proc. NatI. Acad. Sci., 1982.
In 1-d, standard deviation \(\sigma\) scales as
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\sigma \sim t^{\alpha}
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Extensive memory of path now matters...
 collapse
of Reality Mountains


The Road of Surprises
through the Disconcerting Woods


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\section*{Neural reboot (NR):}
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PoCS | @poesvox

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Power-Law
Mechanisms, Pt. 1

\section*{Desert rain frog/Squeaky toy:}

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\section*{Variable Transformation}

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\section*{Understand power laws as arising from}

Elementary distributions (e.g., exponentials).


Random variable \(X\) with known distribution \(P\)

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\section*{Variable Transformation}

Understand power laws as arising from
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\section*{Variable Transformation}

Understand power laws as arising from
1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

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2. Variables connected by power relationships.

Random variable \(X\) with known distribution \(P_{x}\)
Second random variable \(Y\) with \(y=f(x)\).

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\section*{Variable Transformation}

\section*{Understand power laws as arising from}
1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

Random variable \(X\) with known distribution \(P_{x}\)
Second random variable \(Y\) with \(y=f(x)\).
\[
\begin{aligned}
& P_{Y}(y) \mathrm{d} y= \\
& \sum_{x \mid f(x)=y} P_{X}(x) \mathrm{d} x \\
& =\sum_{y \mid f(x)=y} P_{X}\left(f^{-1}(y)\right)_{\mid f^{\prime}\left(f^{-1}(y) \mid\right.} \\
& \sum_{y} f^{-1}
\end{aligned}
\]

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\section*{Variable Transformation}

\section*{Understand power laws as arising from}
1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

Random variable \(X\) with known distribution \(P_{x}\)
Second random variable \(Y\) with \(y=f(x)\).

B
Often easier to do by hand...

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\[
\begin{aligned}
& P_{Y}(y) \mathrm{d} y= \\
& \sum_{x \mid f(x)=y} P_{X}(x) \mathrm{d} x \\
& = \\
& \sum_{y \mid f(x)=y} P_{X}\left(f^{-1}(y)\right) \frac{\mathrm{d} y}{\left|f^{\prime}\left(f^{-1}(y)\right)\right|}
\end{aligned}
\]
hand...

\section*{General Example}

\section*{Assume relationship between \(x\) and \(y\) is 1-1 Power-law relationship between variables:}

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Power-Law
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\section*{General Example}

Assume relationship between \(x\) and \(y\) is 1-1.

\section*{General Example}

Assume relationship between \(x\) and \(y\) is 1-1.
Power-law relationship between variables:
\[
y=c x^{-\alpha}, \alpha>0
\]

PoCs 1 @poesvox

\section*{Random Walks}

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\section*{General Example}

Assume relationship between \(x\) and \(y\) is 1-1.

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\section*{General Example}

Assume relationship between \(x\) and \(y\) is 1-1.
- Power-law relationship between variables:
\[
y=c x^{-\alpha}, \alpha>0
\]

Look at \(y\) large and \(x\) small
\[
\mathrm{d} y=\mathrm{d}\left(c x^{-\alpha}\right)
\]

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\section*{Random Walks}

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\section*{General Example}

Assume relationship between \(x\) and \(y\) is 1-1.
Power-law relationship between variables:
\[
y=c x^{-\alpha}, \alpha>0
\]

Look at \(y\) large and \(x\) small
\[
\begin{gathered}
\mathrm{d} y=\mathrm{d}\left(c x^{-\alpha}\right) \\
=c(-\alpha) x^{-\alpha-1} \mathrm{~d} x
\end{gathered}
\]

Pocs | @poesvox

\section*{Random Walks}

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Assume relationship between \(x\) and \(y\) is 1-1.

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References
\[
=c(-\alpha) x^{-\alpha-1} \mathrm{~d} x
\]
\[
\text { invert: } \mathrm{d} x=\frac{-1}{c \alpha} x^{\alpha+1} \mathrm{~d} y
\]

Assume relationship between \(x\) and \(y\) is 1-1.

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\[
=c(-\alpha) x^{-\alpha-1} \mathrm{~d} x
\]
\[
\begin{aligned}
& \text { invert: } \mathrm{d} x=\frac{-1}{c \alpha} x^{\alpha+1} \mathrm{~d} y \\
& \mathrm{~d} x=\frac{-1}{c \alpha}\left(\frac{y}{c}\right)^{-(\alpha+1) / \alpha} \mathrm{d} y
\end{aligned}
\]

Assume relationship between \(x\) and \(y\) is 1-1.
R Power-law relationship between variables:
\[
y=c x^{-\alpha}, \alpha>0
\]

R Look at \(y\) large and \(x\) small
\[
\begin{gathered}
\mathrm{d} y=\mathrm{d}\left(c x^{-\alpha}\right) \\
\qquad c(-\alpha) x^{-\alpha-1} \mathrm{~d} x \\
\text { invert: } \mathrm{d} x=\frac{-1}{c \alpha} x^{\alpha+1} \mathrm{~d} y \\
\mathrm{~d} x=\frac{-1}{c \alpha}\left(\frac{y}{c}\right)^{-(\alpha+1) / \alpha} \mathrm{d} y \\
\mathrm{~d} x=\frac{-c^{1 / \alpha}}{\alpha} y^{-1-1 / \alpha} \mathrm{d} y
\end{gathered}
\]

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\section*{Now make transformation:}
\[
P_{y}(y) \mathrm{d} y=P_{x}(x) \mathrm{d} x
\]

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Mechanisms, Pt. 1

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\section*{Now make transformation:}
\[
\begin{gathered}
P_{y}(y) \mathrm{d} y=P_{x}(x) \mathrm{d} x \\
P_{y}(y) \mathrm{d} y=P_{x} \overbrace{\left(\left(\frac{y}{c}\right)^{-1 / \alpha}\right)}^{(x)} \overbrace{\frac{c^{1 / \alpha}}{\alpha} y^{-1-1 / \alpha} \mathrm{d} y}^{\mathrm{d} x}
\end{gathered}
\]

PoCs 1 @poesvox
Power-Law
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\section*{Now make transformation:}

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R If \(P_{x}(x) \rightarrow\) non-zero constant as \(x \rightarrow 0\) then
\[
P_{x}(y) \propto y^{-1-1 / \alpha} \text { as } y \rightarrow \infty .
\]

\section*{Now make transformation:}
\[
P_{y}(y) \mathrm{d} y=P_{x}(x) \mathrm{d} x
\]
\[
P_{y}(y) \mathrm{d} y=P_{x} \overbrace{\left(\left(\frac{y}{c}\right)^{-1 / \alpha}\right)}^{(x)} \overbrace{\frac{c^{1 / \alpha}}{\alpha} y^{-1-1 / \alpha} \mathrm{d} y}^{\mathrm{d} x}
\]

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If \(P_{x}(x) \rightarrow\) non-zero constant as \(x \rightarrow 0\) then
\[
P_{x}(y) \propto y^{-1-1 / \alpha} \text { as } y \rightarrow \infty .
\]

If \(P_{x}(x) \rightarrow x^{\beta}\) as \(x \rightarrow 0\) then
\[
P_{y}(y) \propto y^{-1-1 / \alpha-\beta / \alpha} \text { as } y \rightarrow \infty .
\]

\section*{Example}

Random Walks
The First Return Problem Examples

\section*{Exponential distribution}

Given \(P_{x}(x)=\frac{1}{\lambda} e^{-x / \lambda}\) and \(y=c x^{-\alpha}\), then
\[
P(y) \propto y^{-1-1 / \alpha}+O\left(y^{-1-2 / \alpha}\right)
\]

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\section*{Example}

\section*{Exponential distribution}

Given \(P_{x}(x)=\frac{1}{\lambda} e^{-x / \lambda}\) and \(y=c x^{-\alpha}\), then
\[
P(y) \propto y^{-1-1 / \alpha}+O\left(y^{-1-2 / \alpha}\right)
\]

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Exponentials arise from randomness (easy)...

\section*{Example}

\section*{Exponential distribution}

Given \(P_{x}(x)=\frac{1}{\lambda} e^{-x / \lambda}\) and \(y=c x^{-\alpha}\), then
\[
P(y) \propto y^{-1-1 / \alpha}+O\left(y^{-1-2 / \alpha}\right)
\]

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Exponentials arise from randomness (easy)... More later when we cover robustness.

\section*{Outline}

\title{
Variable transformation
}

\section*{Holtsmark's Distribution}


\section*{Gravity}

PoCs 1 @poesvox
Power-Law
Mechanisms, Pt. 1

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\section*{Select a random point in the universe \(\vec{x}\)}

Measure the force of gravity


Holtsmark's Distribution PLIPLO

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\section*{Gravity}

PoCS | @poesvox

Select a random point in the universe \(\vec{x}\)
Measure the force of gravity \(F(\vec{x})\)


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\section*{Gravity}

Select a random point in the universe \(\vec{x}\)
Measure the force of gravity \(F(\vec{x})\)
Observe that \(P_{F}(F) \sim F^{-5 / 2}\).


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PoCS
Principles of
Complex Systems @pocsvox
What's the Story?

\section*{Matter is concentrated in stars: \({ }^{[10]}\)}

\section*{\(F\) is distributed unevenly}


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\section*{Matter is concentrated in stars：\({ }^{[10]}\)}
\(F\) is distributed unevenly
Probability of being a distance \(r\) from a single star at \(\vec{x}=\overrightarrow{0}\) ：
\[
P_{r}(r) \mathrm{d} r \propto r^{2} \mathrm{~d} r
\]
 （oops？） Ans suitura Law of gravity： Connect differentials：dr Connect anneremials． \(\mathrm{dr} e \mathrm{~d} I\)

PoCs｜＠poesvox

Mechanisms，Pt． 1

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\section*{Matter is concentrated in stars：\({ }^{[10]}\)}
\＆\(F\) is distributed unevenly
Probability of being a distance \(r\) from a single star at \(\vec{x}=\overrightarrow{0}\) ：
\[
P_{r}(r) \mathrm{d} r \propto r^{2} \mathrm{~d} r
\]

Assume stars are distributed randomly in space （oops？）


PoCs｜＠poesvox

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\section*{Matter is concentrated in stars: \({ }^{[10]}\)}
\& \(F\) is distributed unevenly
Probability of being a distance \(r\) from a single star at \(\vec{x}=\overrightarrow{0}\) :
\[
P_{r}(r) \mathrm{d} r \propto r^{2} \mathrm{~d} r
\]

Assume stars are distributed randomly in space (oops?)
Assume only one star has significant effect at \(\vec{x}\).

PoCs | @poesvox

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\section*{Matter is concentrated in stars: \({ }^{[10]}\)}
\& \(F\) is distributed unevenly
Probability of being a distance \(r\) from a single star at \(\vec{x}=\overrightarrow{0}\) :
\[
P_{r}(r) \mathrm{d} r \propto r^{2} \mathrm{~d} r
\]
. Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at \(\vec{x}\).

Law of gravity:
\[
F \propto r^{-2}
\]

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\section*{Matter is concentrated in stars: \({ }^{[10]}\)}

PoCs |@poesvox
\(F\) is distributed unevenly
R Probability of being a distance \(r\) from a single star at \(\vec{x}=\overrightarrow{0}\) :
\[
P_{r}(r) \mathrm{d} r \propto r^{2} \mathrm{~d} r
\]

Assume stars are distributed randomly in space (oops?)
Assume only one star has significant effect at \(\vec{x}\).
Law of gravity:
\[
F \propto r^{-2}
\]
. invert:
\[
r \propto F^{-\frac{1}{2}}
\]

\section*{Matter is concentrated in stars: \({ }^{[10]}\)}

PoCs | @poesvox
\(F\) is distributed unevenly
R Probability of being a distance \(r\) from a single star at \(\vec{x}=\overrightarrow{0}\) :
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P_{r}(r) \mathrm{d} r \propto r^{2} \mathrm{~d} r
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Assume stars are distributed randomly in space (oops?)
Assume only one star has significant effect at \(\vec{x}\).
Law of gravity:
\[
F \propto r^{-2}
\]
. invert:
\[
r \propto F^{-\frac{1}{2}}
\]

Connect differentials: \(\mathrm{d} r \propto \mathrm{~d} F^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} \mathrm{~d} F\)

\section*{Transformation:}

PoCs 1 @poesvox
Power-Law
Mechanisms, Pt. 1
Using \(r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F\), and \(P_{r}(r) \propto r^{2}\)

\section*{Random Walks}

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\section*{Transformation:}

PoCS |@poesvox
Power-Law
Mechanisms, Pt. 1
Using \(r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F\), and \(P_{r}(r) \propto r^{2}\) 8
\[
P_{F}(F) \mathrm{d} F=P_{r}(r) \mathrm{d} r
\]

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\section*{Transformation:}

PoCs | @poesvox
Power-Law
Mechanisms, Pt. 1
Using \(r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F\), and \(P_{r}(r) \propto r^{2}\)
8
\[
P_{F}(F) \mathrm{d} F=P_{r}(r) \mathrm{d} r
\]
\[
\propto P_{r}\left(\text { const } \times F^{-1 / 2}\right) F^{-3 / 2} \mathrm{~d} F
\]

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\section*{Transformation:}

PoCs |@poesvox
Power-Law
Mechanisms, Pt. 1
Using \(r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F\), and \(P_{r}(r) \propto r^{2}\) os
\[
P_{F}(F) \mathrm{d} F=P_{r}(r) \mathrm{d} r
\]
\[
\propto P_{r}\left(\text { const } \times F^{-1 / 2}\right) F^{-3 / 2} \mathrm{~d} F
\]
\[
\propto\left(F^{-1 / 2}\right)^{2} F^{-3 / 2} \mathrm{~d} F
\]

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\section*{Transformation:}

PoCs 1 @poesvox
Power-Law
Mechanisms, Pt. 1
Using \(r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F\), and \(P_{r}(r) \propto r^{2}\) B
\[
\begin{gathered}
P_{F}(F) \mathrm{d} F=P_{r}(r) \mathrm{d} r \\
\propto P_{r}\left(\text { const } \times F^{-1 / 2}\right) F^{-3 / 2} \mathrm{~d} F \\
\propto\left(F^{-1 / 2}\right)^{2} F^{-3 / 2} \mathrm{~d} F \\
=F^{-1-3 / 2} \mathrm{~d} F
\end{gathered}
\]

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\section*{Transformation:}

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Mechanisms, Pt. 1
Using \(r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F\), and \(P_{r}(r) \propto r^{2}\) B
\[
\begin{gathered}
P_{F}(F) \mathrm{d} F=P_{r}(r) \mathrm{d} r \\
\propto P_{r}\left(\text { const } \times F^{-1 / 2}\right) F^{-3 / 2} \mathrm{~d} F \\
\propto\left(F^{-1 / 2}\right)^{2} F^{-3 / 2} \mathrm{~d} F \\
=F^{-1-3 / 2} \mathrm{~d} F \\
=F^{-5 / 2} \mathrm{~d} F
\end{gathered}
\]

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\section*{Gravity:}

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\[
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F
\]

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\section*{Gravity:}
\[
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F
\]

8
\[
\gamma=5 / 2
\]

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\section*{Gravity:}
\[
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F
\]
\[
\gamma=5 / 2
\]

\section*{Random Walks}

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\section*{Mean is finite.}

vinw \(\left|\begin{array}{l}0 \\ b \\ 0\end{array}\right|\)
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\section*{Gravity:}
\[
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F
\]
\[
\gamma=5 / 2
\]

\section*{Random Walks}

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\section*{Mean is finite. \\ Rariance \(=\infty\).}

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\section*{Gravity:}
\[
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F
\]

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\[
\gamma=5 / 2
\] PLIPLO

References

\section*{Mean is finite. \\ \& Variance \(=\infty\). \\ A wild distribution.}


\section*{Gravity:}
\[
\begin{gathered}
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F \\
\gamma=5 / 2
\end{gathered}
\]

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(8. Mean is finite.

Q Variance \(=\infty\).
A wild distribution.
Upshot: Random sampling of space usually safe but can end badly...

\section*{Doctorin' the Tardis}

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\section*{\(\square\) Todo: Build Dalek army.}


\section*{Outline}

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\section*{Variable transformation}

\section*{Holtsmark's Distribution}

\section*{PLIPLO}

\section*{Extreme Caution!}

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\section*{PLIPLO = Power law in, power law out}


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\section*{Extreme Caution!}
- PLIPLO = Power law in, power law out

Explain a power law as resulting from another unexplained power law.

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\section*{Extreme Caution!}
\& PLIPLO = Power law in, power law out
Explain a power law as resulting from another unexplained power law.
Yet another homunculus argument \(C^{\top}\)...

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\section*{Extreme Caution!}

Random Walks
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R PLIPLO = Power law in, power law out
Explain a power law as resulting from another unexplained power law.
Yet another homunculus argument \(C\)...
Don't do this!!! (slap, slap)

\section*{Extreme Caution!}

Random Walks
The First Return Problem Examples
R PLIPLO = Power law in, power law out
Explain a power law as resulting from another unexplained power law.
\& Yet another homunculus argument \(\square^{3}\)...
Don't do this!!! (slap, slap)
MIWO = Mild in, Wild out is the stuff.

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\section*{Extreme Caution!}

Random Walks
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R PLIPLO = Power law in, power law out
Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument \(C^{3}\)...

Don't do this!!! (slap, slap)
MIWO = Mild in, Wild out is the stuff.
In general: We need mechanisms!

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\section*{Neural reboot (NR):}
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Pocs | @poesvox

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Power-Law
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\section*{Zoomage in slow motion}

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