

# Mechanisms for Generating Power-Law Size Distributions, Part 1

Principles of Complex Systems | @pocsvox  
CSYS/MATH 300, Fall, 2017

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Vermont Advanced Computing Core | University of Vermont



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Power-Law Mechanisms, Pt. 1

Random Walks  
The First Return Problem  
Examples  
Variable transformation  
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## Outline

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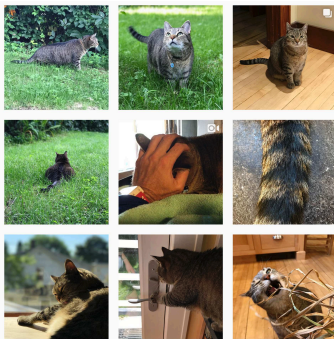
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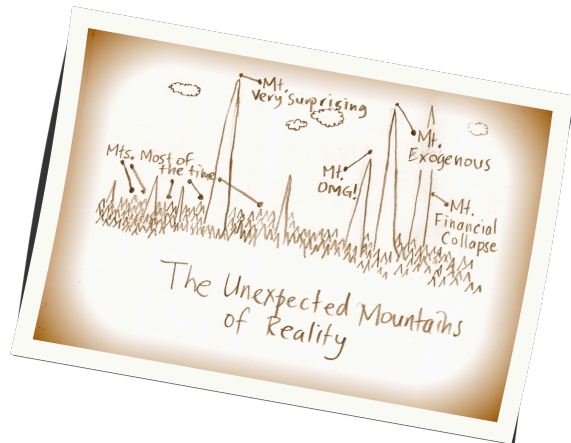
On Instagram at [pratchett\\_the\\_cat](https://www.instagram.com/pratchett_the_cat)

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## Mechanisms:

A powerful story in the rise of complexity:

- 🔗 structure arises out of randomness.
- 🔗 Exhibit A: Random walks. [↗](#)

The essential random walk:

- 🔗 One spatial dimension.
- 🔗 Time and space are discrete
- 🔗 Random walker (e.g., a drunk) starts at origin  $x = 0$ .
- 🔗 Step at time  $t$  is  $\epsilon_t$ :

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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Variances sum: [↗\\*](#)

$$\begin{aligned} \text{Var}(x_t) &= \text{Var}\left(\sum_{i=1}^t \epsilon_i\right) \\ &= \sum_{i=1}^t \text{Var}(\epsilon_i) = \sum_{i=1}^t 1 = t \end{aligned}$$

\* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

- 🔗 A non-trivial scaling law arises out of additive aggregation or accumulation.

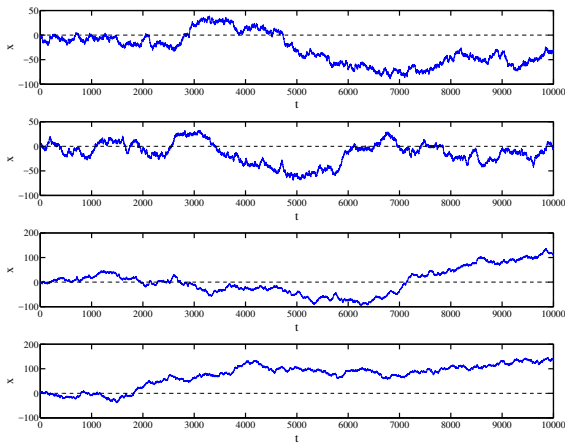
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A few random random walks:



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Random walk basics:

Counting random walks:

- 🔗 Each **specific** random walk of length  $t$  appears with a chance  $1/2^t$ .
- 🔗 We'll be more interested in how many random walks end up at the same place.
- 🔗 Define  $N(i, j, t)$  as # distinct walks that start at  $x = i$  and end at  $x = j$  after  $t$  time steps.
- 🔗 Random walk must displace by  $+(j - i)$  after  $t$  steps.
- 🔗 [Insert question from assignment 3 ↗](#)

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

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Random walks:

Displacement after  $t$  steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \langle \epsilon_i \rangle = 0$$

- 🔗 At any time step, we 'expect' our drunkard to be back at the pub.
- 🔗 Obviously fails for odd number of steps...
- 🔗 But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

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How does  $P(x_t)$  behave for large  $t$ ?

- 🔗 Take time  $t = 2n$  to help ourselves.
- 🔗  $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- 🔗  $x_{2n}$  is even so set  $x_{2n} = 2k$ .
- 🔗 Using our expression  $N(i, j, t)$  with  $i = 0, j = 2k$ , and  $t = 2n$ , we have

$$\Pr(x_{2n} = 2k) \propto \binom{2n}{n+k}$$

- 🔗 For large  $n$ , the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\Pr(x_t = x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

[Insert question from assignment 3 ↗](#)

- 🔗 The whole is different from the parts. **#nutritious**
- 🔗 See also: [Stable Distributions ↗](#)

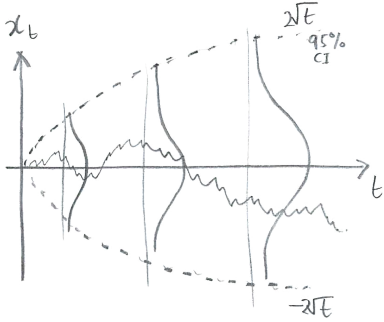
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# Universality is also not left-handed:



- This is Diffusion: the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.

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# Random walks are even weirder than you might think...

- $\xi_{r,t}$  = the probability that by time step  $t$ , a random walk has crossed the origin  $r$  times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact:  $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$
- Even crazier:  
The expected time between tied scores =  $\infty$

See Feller, Intro to Probability Theory, Volume I [3]

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# Applied knot theory:



"Designing tie knots by random walks"  
Fink and Mao,  
Nature, 398, 31-32, 1999. [4]

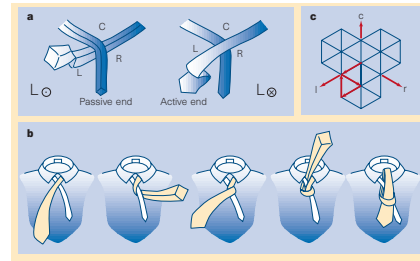


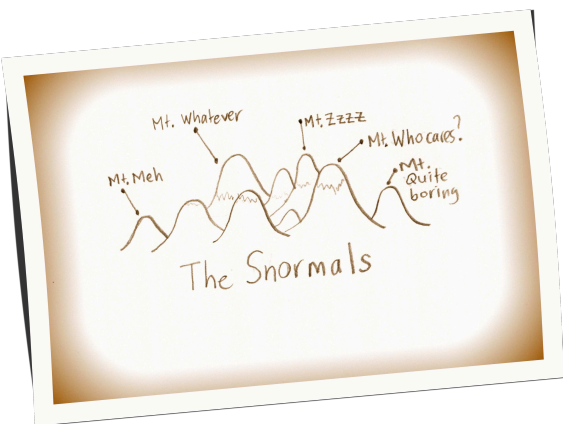
Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie. a. The two ways of beginning a knot, L and R. For knots beginning with L, the tie must begin inside-out. b. The four-in-hand, denoted by the sequence L<sub>0</sub>R<sub>1</sub>L<sub>2</sub>C<sub>3</sub>T<sub>4</sub>. A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk 1111a.

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# Applied knot theory:

$h$	$\gamma$	$\gamma/h$	$K(h, \gamma)$	$s$	$b$	Name	Sequence
3	1	0.33	1	0	0		L <sub>0</sub> R <sub>1</sub> C <sub>2</sub> T
4	1	0.25	1	-1	1	Four-in-hand	L <sub>0</sub> R <sub>1</sub> L <sub>2</sub> C <sub>3</sub> T
5	2	0.40	2	-1	0	Pratt knot	L <sub>0</sub> C <sub>1</sub> R <sub>2</sub> L <sub>3</sub> C <sub>4</sub> T
6	2	0.33	4	0	0	Half-Windsor	L <sub>0</sub> R <sub>1</sub> C <sub>2</sub> L <sub>3</sub> R <sub>4</sub> C <sub>5</sub> T
7	2	0.29	6	-1	1		L <sub>0</sub> R <sub>1</sub> L <sub>2</sub> C <sub>3</sub> R <sub>4</sub> L <sub>5</sub> C <sub>6</sub> T
7	3	0.43	4	0	1		L <sub>0</sub> C <sub>1</sub> R <sub>2</sub> C <sub>3</sub> L <sub>4</sub> R <sub>5</sub> C <sub>6</sub> T
8	2	0.25	8	0	2		L <sub>0</sub> R <sub>1</sub> L <sub>2</sub> C <sub>3</sub> R <sub>4</sub> L <sub>5</sub> R <sub>6</sub> C <sub>7</sub> T
8	3	0.38	12	-1	0	Windsor	L <sub>0</sub> C <sub>1</sub> R <sub>2</sub> L <sub>3</sub> C <sub>4</sub> R <sub>5</sub> L <sub>6</sub> C <sub>7</sub> T
9	4	0.44	8	-1	2		L <sub>0</sub> C <sub>1</sub> R <sub>2</sub> C <sub>3</sub> L <sub>4</sub> C <sub>5</sub> R <sub>6</sub> L <sub>7</sub> C <sub>8</sub> T

Knots are characterized by half-winding number  $h$ , centre number  $\gamma$ , centre fraction  $\gamma/h$ , knots per class  $K(h, \gamma)$ , symmetry  $s$ , balance  $b$ , name and sequence.

- $h$  = number of moves
- $\gamma$  = number of center moves
- $K(h, \gamma) = \frac{2^{\gamma-1}(h-\gamma-2)}{\gamma-1}$
- $s = \sum_{i=1}^h x_i$  where  $x = -1$  for L and +1 for R.
- $b = \frac{1}{2} \sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$  where  $\omega = \pm 1$  represents winding direction.

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# Random walks #crazytownbananapants

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## The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after  $t$  steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

## Reasons for caring:

- We will find a power-law size distribution with an interesting exponent.
- Some physical structures may result from random walks.
- We'll start to see how different scalings relate to each other.

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# Counting first returns:

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## Approach:

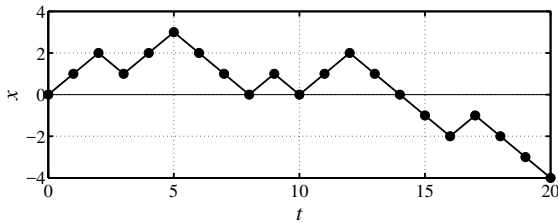
- Move to counting numbers of walks.
- Return to probability at end.
- Again,  $N(i, j, t)$  is the # of possible walks between  $x = i$  and  $x = j$  taking  $t$  steps.
- Consider all paths starting at  $x = 1$  and ending at  $x = 1$  after  $t = 2n - 2$  steps.
- Idea: If we can compute the number of walks that hit  $x = 0$  at least once, then we can subtract this from the total number to find the ones that maintain  $x \geq 1$ .
- Call walks that drop below  $x = 1$  excluded walks.
- We'll use a method of images to identify these excluded walks.

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## For random walks in 1-d:



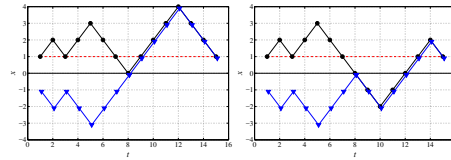
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## Examples of excluded walks:



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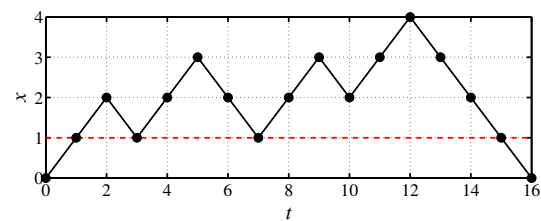


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- A return to origin can only happen when  $t = 2n$ .
- In example above, returns occur at  $t = 8, 10,$  and  $14$ .
- Call  $P_{fr}(2n)$  the probability of first return at  $t = 2n$ .
- Probability calculation  $\equiv$  Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

## Key observation for excluded walks:

- For any path starting at  $x=1$  that hits 0, there is a unique matching path starting at  $x=-1$ .
- Matching path first mirrors and then tracks after first reaching  $x=0$ .
- # of  $t$ -step paths starting and ending at  $x=1$  and hitting  $x=0$  at least once = # of  $t$ -step paths starting at  $x=-1$  and ending at  $x=1 = N(-1, 1, t)$
- So  $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$



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# Probability of first return:

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## Insert question from assignment 3:

- Find

$$N_{fr}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}$$

- Normalized number of paths gives probability.
- Total number of possible paths =  $2^{2n}$ .

$$P_{fr}(2n) = \frac{1}{2^{2n}} N_{fr}(2n) \approx \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}} = \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}$$

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- Can assume drunkard first lurches to  $x = 1$ .
- Observe walk first returning at  $t = 16$  stays at or above  $x = 1$  for  $1 \leq t \leq 15$  (dashed red line).
- Now want walks that can return many times to  $x = 1$ .
- $P_{fr}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- The  $\frac{1}{2}$  accounts for  $x_{2n} = 2$  instead of 0.
- The 2 accounts for drunkards that first lurch to  $x = -1$ .



- We have  $P(t) \propto t^{-3/2}$ ,  $\gamma = 3/2$ .
- Same scaling holds for continuous space/time walks.
- $P(t)$  is normalizable.
- Recurrence:** Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral:** Repeated gambling against an infinitely wealthy opponent must lead to ruin.

### Higher dimensions

- Walker in  $d = 2$  dimensions must also return
- Walker may not return in  $d \geq 3$  dimensions
- Associated genius: [George Pólya](#)

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## Scheidegger networks

- Creates basins with random walk boundaries.
- Observe** that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length  $\ell$  distribution:  $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to  $P(\ell) \propto \ell^{-\gamma}$ .

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## Random walks

### On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the **Invariant Density** of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

### On networks:

- On networks, a random walker visits each node with frequency  $\propto$  node degree #groovy
- Equal probability still present: walkers traverse **edges** with equal frequency. #totallygroovy

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## Connections between exponents:

- For a basin of length  $\ell$ , width  $\propto \ell^{1/2}$
- Basin area  $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert:  $\ell \propto a^{2/3}$
- $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$
- Pr(basin area =  $a$ )**  
 $= \mathbf{Pr}(\text{basin length} = \ell) d\ell$   
 $\propto \ell^{-3/2} d\ell$   
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$   
 $= a^{-4/3} da$   
 $= a^{-\tau} da$

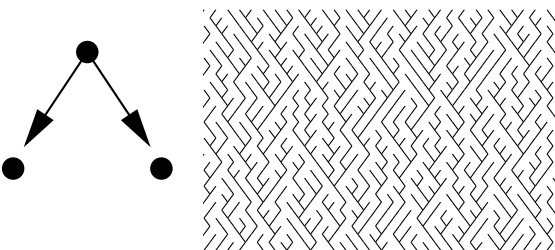
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## Scheidegger Networks [9, 2]



- Random directed network on triangular lattice.
- Toy model of real networks.
- 'Flow' is southeast or southwest with equal probability.

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## Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly:  $1.3 < \tau < 1.5$  and  $1.5 < \gamma < 2$

### Generalize relationship between area and length:

- Hack's law [5]:  $\ell \propto a^h$ .

- For real, large networks  $h \simeq 0.5$
- Smaller basins possibly  $h > 1/2$  (see: allometry).
- Models exist with interesting values of  $h$ .
- Plan:** Redo calc with  $\gamma$ ,  $\tau$ , and  $h$ .

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## Connections between exponents:

Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

$d\ell \propto d(a^h) = ha^{h-1}da$

Find  $\tau$  in terms of  $\gamma$  and  $h$ .

$$\begin{aligned} \Pr(\text{basin area} = a)da &= \Pr(\text{basin length} = \ell)d\ell \\ &\propto \ell^{-\gamma}d\ell \\ &\propto (a^h)^{-\gamma}a^{h-1}da \\ &= a^{-(1+h(\gamma-1))}da \end{aligned}$$

$$\tau = 1 + h(\gamma - 1)$$

Excellent example of the **Scaling Relations** found between exponents describing power laws for many systems.

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## More than randomness

Can generalize to Fractional Random Walks [7, 8, 6]

Levy flights, Fractional Brownian Motion

See Montroll and Shlesinger for example: [6]  
"On  $1/f$  noise and other distributions with long tails."

Proc. Natl. Acad. Sci., 1982.

In 1-d, standard deviation  $\sigma$  scales as

$$\sigma \sim t^\alpha$$

$\alpha = 1/2$  — diffusive

$\alpha > 1/2$  — superdiffusive

$\alpha < 1/2$  — subdiffusive

Extensive memory of path now matters...

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## Connections between exponents:

With more detailed description of network structure,  $\tau = 1 + h(\gamma - 1)$  simplifies to: [1]

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$

Only one exponent is independent (take  $h$ ).

Simplifies system description.

Expect Scaling Relations where power laws are found.

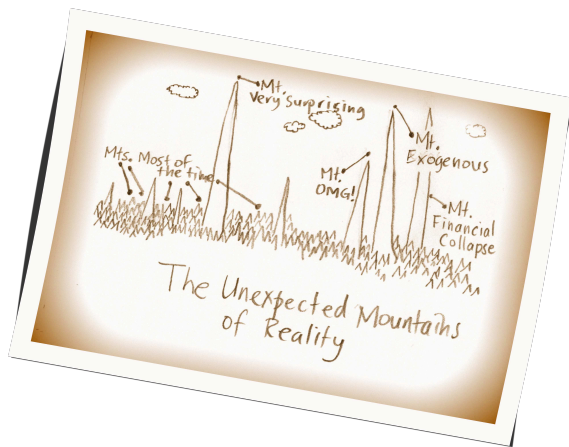
Need only characterize Universality class with independent exponents.

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## Other First Returns or First Passage Times:

### Failure:

A very simple model of failure/death: [11]

$x_t$  = entity's 'health' at time  $t$

Start with  $x_0 > 0$ .

Entity fails when  $x$  hits 0.

### Streams

Dispersion of suspended sediments in streams.

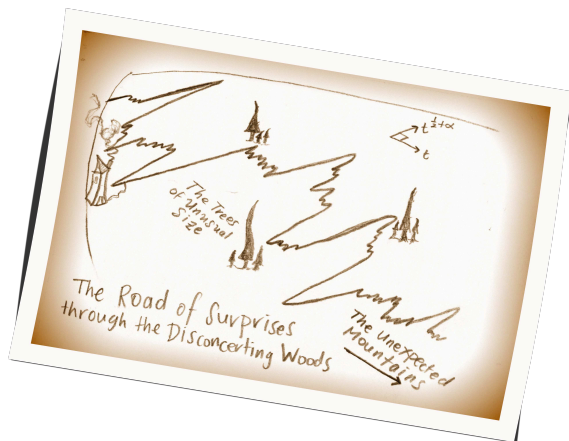
Long times for clearing.

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# Variable Transformation

## Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- ☞ Random variable  $X$  with known distribution  $P_x$
- ☞ Second random variable  $Y$  with  $y = f(x)$ .

$$P_Y(y)dy = \sum_{x|f(x)=y} P_X(x)dx = \sum_{y|f(x)=y} P_X(f^{-1}(y)) \left| \frac{dy}{f'(f^{-1}(y))} \right|$$

- ☞ Often easier to do by hand...

## General Example

- ☞ Assume relationship between  $x$  and  $y$  is 1-1.
- ☞ Power-law relationship between variables:  $y = cx^{-\alpha}$ ,  $\alpha > 0$
- ☞ Look at  $y$  large and  $x$  small

$$dy = d(cx^{-\alpha}) = c(-\alpha)x^{-\alpha-1}dx$$

$$\text{invert: } dx = \frac{-1}{c\alpha} x^{\alpha+1} dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-\frac{\alpha+1}{\alpha}} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$

## Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left( \left(\frac{y}{c}\right)^{-1/\alpha} \right) \frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$

- ☞ If  $P_x(x) \rightarrow$  non-zero constant as  $x \rightarrow 0$  then

$$P_x(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- ☞ If  $P_x(x) \rightarrow x^\beta$  as  $x \rightarrow 0$  then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

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# Example

## Exponential distribution

Given  $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$

- ☞ Exponentials arise from randomness (easy)...
- ☞ More later when we cover robustness.

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# Gravity

- ☞ Select a random point in the universe  $\vec{x}$
- ☞ Measure the force of gravity  $F(\vec{x})$
- ☞ Observe that  $P_F(F) \sim F^{-5/2}$ .



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## Matter is concentrated in stars: [10]

- ☞  $F$  is distributed unevenly
- ☞ Probability of being a distance  $r$  from a single star at  $\vec{x} = \vec{0}$ :

$$P_r(r)dr \propto r^2 dr$$

- ☞ Assume stars are distributed randomly in space (oops?)
- ☞ Assume only one star has significant effect at  $\vec{x}$ .
- ☞ Law of gravity:

$$F \propto r^{-2}$$

- ☞ invert:

$$r \propto F^{-1/2}$$

- ☞ Connect differentials:  $dr \propto dF^{-1/2} \propto F^{-3/2} dF$

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## Transformation:

Using  $r \propto F^{-1/2}$ ,  $dr \propto F^{-3/2}dF$ , and  $P_r(r) \propto r^{-2}$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF$$



$$\propto (F^{-1/2})^2 F^{-3/2}dF$$



$$= F^{-1-3/2}dF$$



$$= F^{-5/2}dF.$$

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## Extreme Caution!

- PLIPLD = **Power law in, power law out**
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument [↗](#)...
- Don't do this!!! (slap, slap)
- MIWO = **Mild in, Wild out** is the stuff.
- In general: We need mechanisms!

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## Gravity:

$$P_F(F) = F^{-5/2}dF$$



$$\gamma = 5/2$$

- Mean is finite.
- Variance =  $\infty$ .
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

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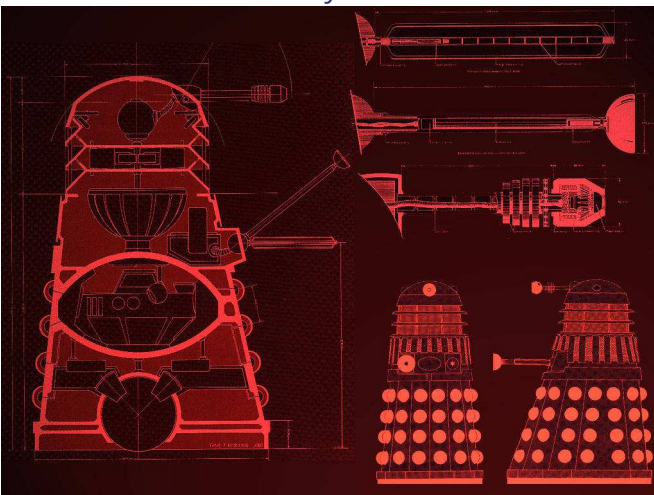
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☐ Todo: Build Dalek army.



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