1. (Q 4, 6.5) Show that the function \( f(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2 \) does not have a minimum at \((0, 0)\) even though it has positive coefficients.

Do this by rewriting \( f(x_1, x_2) \) as \( f(x_1, x_2) = x_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) and finding the pivots of \( A \) and noting their signs (and explaining why the signs of the pivots matter).

Write \( f \) as a difference of squares and find a point \((x_1, x_2)\) where \( f \) is negative.

Note of caution: All of this signs matching for pivots and eigenvalues falls apart if we have to do row swaps in our reduction.

**Solution:**

First, we can rewrite our function as

\[
\begin{bmatrix} x_1 & x_2 \end{bmatrix} f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

We need to do one step of row reduction to reveal the pivots:

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \stackrel{R_2 \rightarrow R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.
\]

The pivots are 1 and -1 so we must have one positive and one negative eigenvalue: \( f \) is therefore not positive definite.

Completing the square:

\[
f(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2 = (x_1 + 2x_2)^2 - 4x_2^2 + 3x_2^2 = (x_1 + 2x_2)^2 - 1 \cdot (x_2)^2.
\]

Note the appearance of the pivots 1 and -1 in front of the squares. As we saw in class, the LU factorization of symmetric matrices, \( A = LDL^T \), is behind all of this.

\( \square \)

2. (Q 9, 6.5) Find the 3 by 3 matrix \( A \) and its pivots, rank, eigenvalues, and determinant:

\[
\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.
\]
Is this matrix positive definite, semi-positive definite, or neither?

**Solution:**

Expanding $4(x_1 - x_2 + 2x_3)^2$ we have

$$4x_1^2 + 4x_2^2 + 16x_3^2 - 8x_1x_2 - 16x_2x_3 + 16x_3x_1$$

and this can be written as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$ 

We can now find the pivots of $A$ (much easier than finding the eigenvalues):

$$\begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix} \xrightarrow{R2' = R2 - \frac{1}{4} R1} \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3' = R3 - \frac{1}{2} R1} \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

The pivots are 4, 0, 0 and our matrix is therefore semi-positive definite.

Some bonus sneaky grooviness: we can see straight away that $A$ is a rank one matrix:

$$A = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \end{bmatrix} = 24 \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}.$$ 

We now have $A$ in its spectral decomposition form:

$$A = \sum_{i=1}^{n} \lambda_i \hat{u}_i \hat{v}_i^T.$$ 

So the eigenvalues are 24, 0, and 0, which means that $A$ is semi-positive definite.

Another way to see this: we know from the pivots that two of the eigenvalues are 0. Since the trace of $A$ is the sum of the eigenvalues, we have that the trace of $A$ must be $\lambda_1 + 0 + 0 = \lambda_1$. Checking $A$, we have $\lambda_1 = 24$.

The determinant of $A$ is zilch since we have 0 eigenvalues. 

□

3. (following set of questions based on Q 7, Section 6.7)

Singular Value Decomposition = Happiness.

Consider

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
(a) What are $m$, $n$, and $r$ for this matrix?
(b) What are the dimensions of $U$, $\Sigma$, and $V$?
(c) Calculate $A^TA$ and $AA^T$.

Solution:

(a) $m = 2$, $n = 3$, and $r = 2$.
(b) $U$ is 2x2, $\Sigma$ is 2x3, and $V$ is 3x3.
(c) 
\[
A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]

and

\[
AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}.
\]

4. For the matrix $A$ given above, find the eigenvalues and eigenvectors of $A^TA$, and thereby construct $V$ and $\Sigma$.

See this tweet for some post-it based help:
https://twitter.com/matrixologyvox/status/593540446845947904

Solution:

Okay, we have to solve $|A - \lambda I| = 0$. Using the 'big formula' and going across the top row (to take advantage of the 0 in the (1,3) entry), we have:

\[
0 = \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 1 & 2 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda) \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 - \lambda \end{vmatrix}
\]

\[
= (1 - \lambda)[(2 - \lambda)(1 - \lambda) - (1)(1)] - (1)(1 - \lambda) - (0)(1)
\]

\[
= -\lambda^3 4\lambda^2 - 3\lambda
\]

\[
= -\lambda(\lambda - 3)(\lambda - 1).
\]

Our eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_1 = 0$. Ordering for largest to smallest is important here.

We notice a couple of things: (1) The eigenvalues are all $\geq 0$. This is good as these are the squares of our singular values, the $\sigma_i$. (2) One eigenvalue is 0. This
makes sense as the rank $r = 2$ which means that we have two non-zero singular values.

Our singular values are the square roots of the eigenvalues:

$$\sigma_1 = \sqrt{3} \text{ and } \sigma_2 = 1.$$ 

Note that there are only two singular values as $A$ is 2x3.

Next task: find the eigenvectors.

(a) For $\lambda_1 = 3$, we solve $(A^T A - 3I)\vec{v}_1 = \vec{0}$.

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \vec{v}_1 = \vec{0}.$$ 

You can do this be inspection, or by systematically finding the nullspace vector, or however you please. By inspection: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Normalizing, we have $\hat{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(b) For $\lambda_2 = 1$, we solve $(A^T A - I)\vec{v}_2 = \vec{0}$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{v}_2 = \vec{0}.$$ 

By inspection: $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and the normalized eigenvector is $\hat{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(c) For $\lambda_3 = 0$, solve $(A^T A - 0I)\vec{v}_3 = \vec{0}$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{v}_3 = \vec{0}.$$
By inspection: $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and the normalized eigenvector is $\hat{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

We can now write down $V = [\hat{v}_1 | \hat{v}_2 | \hat{v}_3]$:

\[
V = \begin{bmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
\frac{\sqrt{2}}{6} & 0 & \frac{-1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}}
\end{bmatrix}.
\]

And the $\Sigma$ matrix is

\[
\begin{bmatrix}
\sqrt{3} & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]

5. For the same $A$, now find the basis $\{\hat{u}_i\}$ using the essential connection $A \hat{v}_i = \sigma_i \hat{u}_i$.

Construct $U$ from the basis you find.

Again see this tweet for some post-it based help: https://twitter.com/matrixologyvox/status/593540446845947904

Solution:

We multiply the $\hat{v}_i$ for which $\sigma_i > 0$ by $A$ to find the $\hat{u}_i$. We’ll need to pull the $\sigma_i$ out to find the $\hat{u}_i$. Recall that $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$. First off:

\[
A \hat{v}_1 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
\]

\[
= \frac{1}{\sqrt{6}} \begin{bmatrix}
3 \\
3
\end{bmatrix}
\]

\[
= \sqrt{3} \frac{1}{\sqrt{3} \sqrt{6}} \begin{bmatrix}
3 \\
3
\end{bmatrix}
\]

\[
= \sqrt{3} \frac{1}{3 \sqrt{2}} \begin{bmatrix}
3 \\
3
\end{bmatrix}
\]

\[
= \sqrt{3} \frac{1}{\sqrt{2}} \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

\[
= \sigma_1 \hat{u}_1.
\]
Notice how when we pull out $\sigma_1$, we (almost magically) end up with a happy little unit vector.

Second vector:

$$\mathbf{A}\hat{v}_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 1 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \sigma_2 \hat{u}_2.$$  

Smashing. Note that $\hat{u}_1^T \hat{u}_2 = 0$ and we have an orthonormal basis for $\mathbb{R}^2$.

Finally,

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$  

6. Next find the $\{\hat{u}_i\}$ in a different way by finding the eigenvalues and eigenvectors of $\mathbf{A}\mathbf{A}^T$.

**Solution:**

Eigenvalues:

$$0 = |\mathbf{A}\mathbf{A}^T - \lambda I| = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix}$$

$$= (2 - \lambda)^2 - 1$$

$$= (2 - \lambda - 1)(2 - \lambda + 1)$$

$$= (1 - \lambda)(3 - \lambda),$$

where we have used the difference of perfect squares.

So $\lambda_1 = 3$ and $\lambda_2 = 1$ which again gives $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

Eigenvector time (la-la-la-la) for $\lambda_1 = 3$:

$$\bar{\mathbf{u}} = (\mathbf{A}\mathbf{A}^T - \lambda_1 I)\hat{u}_1$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3I$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$
By inspection, we have \( \hat{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

Next,

\[
\tilde{0} = (\mathbf{A} \mathbf{A}^T - \lambda_2 I) \hat{u}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

This gives \( \hat{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Note that we could have chosen \( \hat{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), the negative of the one we have above.

In fact, we always need to compute \( \mathbf{A} \hat{v}_i \) to find out which direction \( \hat{u}_i \) should take. Beyond this, we don’t need to compute the \( \hat{u}_i \) directly ever as once we have \( \hat{v}_i \) we need only multiply by \( \mathbf{A} \) (as per the previous question). We found the \( u \)'s directly here to (1) see that both ways give the same thing and (2) punish ourselves just a little more.

\[ \square \]

7. (a) Put everything together and show that \( \mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \).

(b) Draw the 'big picture' for this \( \mathbf{A} \) showing which \( \hat{v}_i \)'s are mapped to which \( \hat{u}_i \)'s.

(c) Which basis vectors, if any, belong to the two nullspaces?

**Solution:**

(a)

\[
\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{3} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 + 0 & 1/2 - 1/2 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + 0 & 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.
\]
(c) Left nullspace is just \{\mathbf{0}\}.

$A$’s nullspace has dimension 1 and has the basis vector \( \hat{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \).

8. Finally, for this same $A$, perform the following calculation:

$$\sigma_1 \hat{u}_1 \hat{v}_1^T + \sigma_2 \hat{u}_2 \hat{v}_2^T + \ldots + \sigma_r \hat{u}_r \hat{v}_r^T$$

where $r$ is the rank of $A$. 

\[ \begin{array}{cc}
A & \text{nullspace} \\
\text{col space} & \Rightarrow \\
\text{row space}
\end{array} \]
You should obtain A...

Solution:

\[
\sigma_1 \hat{u}_1 \hat{v}_1^T + \sigma_2 \hat{u}_2 \hat{v}_2^T = \sqrt{3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \mathbf{A}.
\]

\[
\begin{array}{c}
-0.2361 \\
4.2361
\end{array}
\]


Verify the signs you found for the pivots of \( \mathbf{A} \) in question 1 by using Matlab to find \( \mathbf{A} \)'s eigenvalues.

Solution:

Using Matlab, we find \( \lambda_1 = -0.2361 \) and \( \lambda_2 = 4.2361 \):

\[
\text{>> eig([ 1 2; 2 3])}
\]

\[
\text{ans =}
\]

\[
\begin{array}{c}
-0.2361 \\
4.2361
\end{array}
\]

One positive and one negative, matching the signs of the pivots.

\[
\begin{array}{c}
\end{array}
\]

10. Matlab question.

Use Matlab to compute the SVD for the matrix \( \mathbf{A} \) you explored in questions 3–8.

Solution:

\[
\text{>> [U,Sigma,V] = svd([ 1 1 0 ; 0 1 1])}
\]

\[
\text{U =}
\]

\[
\begin{array}{c}
-0.7071 \\
-0.7071
\end{array}
\]

\[
\begin{array}{c}
-0.7071 \\
0.7071
\end{array}
\]

\[
\text{Sigma =}
\]

\[
\begin{array}{c}
\end{array}
\]

\[
\begin{array}{c}
\end{array}
\]
1.7321 0 0
0 1.0000 0

\[ V = \]
-0.4082 -0.7071 0.5774
-0.8165 0.0000 -0.5774
-0.4082 0.7071 0.5774

11. (The bonus one pointer)

Where does the fearsome kiwi rank among rattites and what’s unusual about the kiwi egg?

**Solution:**

The kiwi is the smallest of all struthious birds.

A kiwi egg can weight up to \( \frac{1}{4} \) of the mother’s own weight, which is believed to be the highest ratio of all birds.