Generating Functions and Networks

Complex Networks | @networksvox CSYS/MATH 303, Spring, 2016

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Dept. of Mathematics & Statistics | Vermont Complex Systems Center Vermont Advanced Computing Core | University of Vermont























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COcoNuTS

Generating Functions

Definitions

Giant Componi Condition

Component sizes

Size of the Gia

Average Component Size





These slides are brought to you by:



COCONUTS

Generating

Basic Properties Giant Component Component sizes

Useful results Size of the Giant Component Average Component Size







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A Idea: Given a sequence a_0, a_1, a_2, \dots , associate each element with a distinct function or other mathematical object.

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Idea: Given a sequence $a_0, a_1, a_2, ...$, associate each element with a distinct function or other mathematical object.

Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

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- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Roughly: transforms a vector in R^{∞} into a function defined on R^1 .

Related to Fourier, Laplace, Mellin,

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Rolling dice and flipping coins:

$$p_k^{(2)} = \mathbf{Pr}(\mathsf{throwing}\;\mathsf{a}\;k) = 1/6\;\mathsf{where}\;k = 1, 2, \dots, 6.$$

$$F^{(\bigodot)}(x) = \sum_{k=1}^6 p_k^{(\bigodot)} x^k = \frac{1}{6} (x + x^2 + x^3 + x^4 + x^5 + x^6).$$

$$p_0^{\text{(coin)}} = \mathbf{Pr}(\text{head}) = 1/2, p_1^{\text{(coin)}} = \mathbf{Pr}(\text{tail}) = 1/2$$

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})} x^0 + p_1^{(\text{coin})} x^1 = \frac{1}{2} (1+x)$$

A generating function for a probability distribution is called a Probability Generating Function (p.g.f.).

We'll come back to these simple examples as we derive various delicious properties of generating functions.

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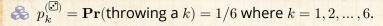
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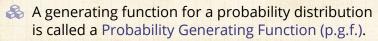
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Take a degree distribution with exponential decay:

$$P_k = c e^{-\lambda k}$$

where geometric sumfully, we have $c=1-e^{-\lambda}$

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Notice that $F(1) = c/(1-e^{-\lambda}) = 1$.

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For probability distributions, we must always have F(1) = 1 since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k$$



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Check die and coin p.g.f.'s.

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Properties:



Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k$$

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Properties:



Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

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Average degree:

$$\begin{split} \langle k \rangle &= \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1} \\ &= \left. \frac{\mathrm{d}}{\mathrm{d}x} F(x) \right|_{x=1} \end{split}$$

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In general, many calculations become simple, if a little abstract.

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- For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.$$



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Properties:

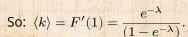
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So:
$$\langle k \rangle = F'(1) = \frac{e^{-\lambda}}{(1-e^{-\lambda})}$$
.

Check for die and coin p.g.f.'s.



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Normalization:

F(1) =

First moment:

 $\langle k \rangle = F'(1$

Higher moments.

 $\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^nP(x)$

kth element of sequence (general):

 $P_k = \frac{1}{k!} \frac{\mathsf{d}^k}{\mathsf{d}x^k} F(x)$

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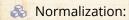
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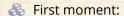








$$F(1) = 1$$



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The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x)F_V(x).$$

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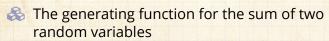
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Insert question from assignment 5 ...

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Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

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COCONUTS

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 $F_{\mathcal{P}}(x)$ is the g.f. for $P_{\mathcal{P}}$. $F_R(x)$ is the g.f. for R_k .

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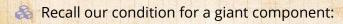
Giant Component Condition

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$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's re-express our condition in terms of generating functions.
- & We first need the g.f. for R_k .
- We'll now use this notation:

$$\frac{F_P(x)}{F_R(x)}$$
 is the g.f. for $\frac{P_k}{R_k}$.

Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F_R'(1) > 1.$$

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Edge-degree distribution

Recall our condition for a giant component:

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Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F_R'(1) > 1.$$

& Now find how F_R is related to F_P ...

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We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

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Shift index to j = k + 1 and pull out $\frac{1}{\langle k \rangle}$:

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$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to j = k + 1 and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k x^k}{k} = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

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$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathsf{d}}{\mathsf{d}x} \sum_{j=1}^{\infty} P_j x^j$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

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$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0 \right)$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

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$$=\frac{1}{\langle k\rangle}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=1}^{\infty}P_{j}x^{j}=\frac{1}{\langle k\rangle}\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{P}(x)-P_{0}\right)=\frac{1}{\langle k\rangle}F_{P}'(x).$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to j = k + 1 and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d} x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0 \right) = \frac{1}{\langle k \rangle} F_P'(x).$$

Finally, since $\langle k \rangle = F_P'(1)$,

$$F_R(x) = \frac{F_P'(x)}{F_P'(1)}$$

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Recall giant component condition is $\langle k \rangle_R = F_R'(1) > 1.$

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- Recall giant component condition is $\langle k \rangle_R = F_R'(1) > 1$.
- Since we have $F_R(x) = F_P'(x)/F_P'(1)$,

Setting | | | 1, | our condition becomes

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- Recall giant component condition is $\langle k \rangle_R = F_R'(1) > 1.$
- Since we have $F_R(x) = F_P'(x)/F_P'(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}$$

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- Recall giant component condition is $\langle k \rangle_R = F_R'(1) > 1$.
- 3 Since we have $F_R(x) = F_P'(x)/F_P'(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}$$

Setting x = 1, our condition becomes

$$\frac{F_P''(1)}{F_P'(1)} > 1$$

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To figure out the size of the largest component (S_1) , we need more resolution on component sizes.

Definitions:



 $\Re \pi_n$ = probability that a random node belongs to a finite component of size $n < \infty$.

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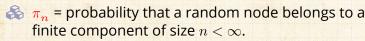






To figure out the size of the largest component (S_1) , we need more resolution on component sizes.

Definitions:



 $\underset{\rho_n}{\lozenge}$ = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$. Generating

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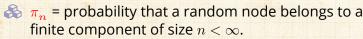


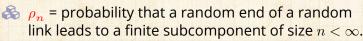
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Size distributions

To figure out the size of the largest component (S_1) , we need more resolution on component sizes.

Definitions:





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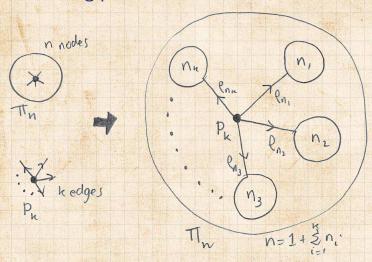
Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$
 neighbors \Leftrightarrow components





Connecting probabilities:



Markov property of random networks connects π_n , ρ_n , and P_k .

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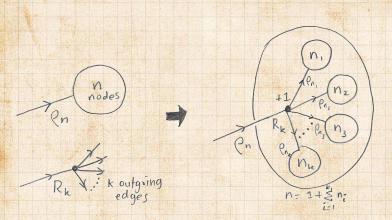
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Connecting probabilities:



 $\ref{eq:sphere:$

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G.f.'s for component size distributions:

$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n$$
 and $F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$

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$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

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$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:



 \Re Subtle key: $F_{\pi}(1)$ is the probability that a node belongs to a finite component.

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$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

Subtle key: $F_{\pi}(1)$ is the probability that a node belongs to a finite component.

 \Leftrightarrow Therefore: $S_1 = 1 - F_{\pi}(1)$.

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$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

- Subtle key: $F_{\pi}(1)$ is the probability that a node belongs to a finite component.
- \Leftrightarrow Therefore: $S_1 = 1 F_{\pi}(1)$.

Our mission, which we accept:

Determine and connect the four generating functions

$$F_P, F_R, F_{\pi}, \text{ and } F_{\rho}.$$

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Sneaky Result 1:

- Consider two random variables *U* and *V* whose values may be 0, 1, 2, ...
- Write probability distributions as U_k and V_k and g(f)'s as F_{m} and F_{k} .
- SR1: If a third random variable is defined as

 $V^{(i)}$ with each $V^{(i)} \stackrel{d}{=} V$

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Sneaky Result 1:

- \triangle Consider two random variables U and V whose values may be 0, 1, 2, ...



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Sneaky Result 1:

- \triangle Consider two random variables U and V whose values may be $0, 1, 2, \dots$
- \triangle Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V .

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Sneaky Result 1:

- \Leftrightarrow Consider two random variables U and V whose values may be 0, 1, 2, ...
- \ref{Model} Write probability distributions as \ref{U}_k and \ref{V}_k and g.f.'s as F_U and F_V .
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{U} V^{(i)}$$
 with each $V^{(i)} \stackrel{d}{=} V$

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Sneaky Result 1:

- \Leftrightarrow Consider two random variables U and V whose values may be 0, 1, 2, ...
- \ref{Model} Write probability distributions as \ref{U}_k and \ref{V}_k and g.f.'s as F_U and F_V .
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{U} V^{(i)}$$
 with each $V^{(i)} \stackrel{d}{=} V$

then

$$F_W(x) = F_U(F_V(x))$$

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$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

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Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j imes ext{Pr(sum of } j ext{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_{j} \sum_{\substack{\{i_{1},i_{2},\ldots,i_{j}\}|\\i_{1}+i_{2}+\ldots+i_{j}=k}} V_{i_{1}}V_{i_{2}}\cdots V_{i_{j}}$$

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Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\stackrel{\{i_1,i_2,\ldots,i_j\}|}{i_1+i_2+\ldots+i_i=k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

References







Proof of SR1:

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$$=\sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty}$$

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Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \operatorname{Pr(sum} \text{ of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1,i_2,\dots,i_j\}|\\ i_1+i_2+\dots+i_j=k}} V_{i_1}V_{i_2}\cdots V_{i_j}$$

$$= \sum_{j=0}^{\infty} \underbrace{U_j}_{\substack{k=0}}^{\infty} \sum_{\substack{\{i_1,i_2,\dots,i_j\}|\\i_1+i_2+\dots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}$$

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$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1,i_2,\dots,i_j\}|\\i_1+i_2+\dots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}$$

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$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1,i_2,\dots,i_j\} |\\ i_1+i_2+\dots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j} \underbrace{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j} = (F_V(x))^j$$

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$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} | \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}$$

$$x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j$$

$$\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j = (F_V(x))^j$$

$$= \sum_{j=0}^{\infty} U_j \left(F_V(x)\right)^j$$

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$$F_W(x) = \sum_{j=0}^\infty U_j \sum_{k=0}^\infty \sum_{\substack{\{i_1,i_2,\dots,i_j\}|\\i_1+i_2+\dots+i_j=k\\}} V_{i_1}x^{i_1}V_{i_2}x^{i_2}\cdots V_{i_j}x^{i_j}$$

$$x^k \text{ piece of } \left(\sum_{i'=0}^\infty V_{i'}x^{i'}\right)^j$$

$$\left(\sum_{i'=0}^\infty V_{i'}x^{i'}\right)^j = (F_V(x))^j$$

$$= \sum_{j=0}^\infty U_j \left(F_V(x)\right)^j$$

$$= F_U \left(F_V(x)\right)$$

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$$F_W(x) = \sum_{j=0}^\infty U_j \sum_{k=0}^\infty \sum_{\substack{\{i_1,i_2,\dots,i_j\}|\\i_1+i_2+\dots+i_j=k\\}} V_{i_1}x^{i_1}V_{i_2}x^{i_2}\cdots V_{i_j}x^{i_j}$$

$$x^k \text{ piece of } \left(\sum_{i'=0}^\infty V_{i'}x^{i'}\right)^j$$

$$\left(\sum_{i'=0}^\infty V_{i'}x^{i'}\right)^j = (F_V(x))^j$$

$$= \sum_{j=0}^\infty U_j \left(F_V(x)\right)^j$$

$$= F_U \left(F_V(x)\right)$$

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Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^\infty U_j \sum_{k=0}^\infty \sum_{\substack{\{i_1,i_2,\dots,i_j\}|\\i_1+i_2+\dots+i_j=k\\}} V_{i_1}x^{i_1}V_{i_2}x^{i_2}\cdots V_{i_j}x^{i_j}$$

$$x^k \text{ piece of } \left(\sum_{i'=0}^\infty V_{i'}x^{i'}\right)^j$$

$$\left(\sum_{i'=0}^\infty V_{i'}x^{i'}\right)^j = (F_V(x))^j$$

$$= \sum_{j=0}^\infty U_j \left(F_V(x)\right)^j$$

$$= F_U \left(F_V(x)\right)$$

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Alternate, groovier proof in the accompanying assignment.

Sneaky Result 2:

- Start with a random variable U with distribution U_k ($k=0,1,2,\ldots$)
- SR2: If a second random variable is defined as

Reason:
$$V_k = U_{k-1}$$
 for $k \ge 1$ and $V_0 = 0$

$$F_V(x) = \sum_{k=0}^{\infty} V_k x^k$$

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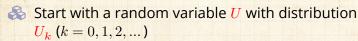
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Sneaky Result 2:



SR2: If a second random variable is defined as

Reason: $V_k = U_{k-1}$ for $k \ge 1$ and $V_0 = 0$

$$F_V(x) = \sum_{k=0}^{\infty} V_k x^k$$

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Sneaky Result 2:

- Start with a random variable U with distribution U_k (k=0,1,2,...)
- SR2: If a second random variable is defined as

$$V = U + 1$$
 then $F_V(x) = xF_D(x)$

Reason: $V_k = U_{k-1}$ for $k \ge 1$ and $V_0 = 0$

$$P_V(x) = \sum_{k=0}^{\infty} V_k x^k$$

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Sneaky Result 2:

- \longrightarrow Start with a random variable U with distribution U_k (k = 0, 1, 2, ...)
- SR2: If a second random variable is defined as

$$V = U + 1$$
 then $F_V(x) = xF_U(x)$

$$F_{V}(x) = xF_{U}(x)$$

$$F_V(x) = \sum_{k=0}^{\infty} V_k x^k$$

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Sneaky Result 2:

- Start with a random variable U with distribution U_k (k=0,1,2,...)
- SR2: If a second random variable is defined as

$$V = U + 1$$
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 \Re Reason: $V_k = U_{k-1}$ for $k \ge 1$ and $V_0 = 0$.

$$F_V(x) = \sum_{k=0}^{\infty} V_k x^k$$

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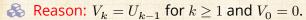




Sneaky Result 2:

- Start with a random variable U with distribution U_k (k=0,1,2,...)
- SR2: If a second random variable is defined as

$$V = U + 1$$
 then $F_V(x) = xF_U(x)$





 $=x\sum^{\infty}U_{j}x^{j}=xF_{U}(x).$



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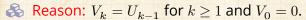




Sneaky Result 2:

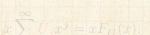
- Start with a random variable U with distribution U_k (k=0,1,2,...)
- SR2: If a second random variable is defined as

$$V = U + 1$$
 then $F_V(x) = xF_U(x)$





$$\dot{\cdot\cdot} F_V(x) = \sum_{k=0}^\infty V_k x^k = \sum_{k=1}^\infty {\color{red}U_{k-1} x^k}$$



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Sneaky Result 2:

- Start with a random variable U with distribution U_k (k=0,1,2,...)
- SR2: If a second random variable is defined as

$$V = U + 1$$
 then $F_V(x) = xF_U(x)$





$$\begin{split} \dot{\cdot} F_V(x) &= \sum_{k=0}^\infty V_k x^k = \sum_{k=1}^\infty \underbrace{U_{k-1} x^k}_{} \\ &= x \sum_{j=0}^\infty \underbrace{U_j x^j}_{} = x F_U(x). \end{split}$$



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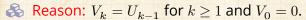




Sneaky Result 2:

- Start with a random variable U with distribution U_k (k=0,1,2,...)
- SR2: If a second random variable is defined as

$$V=U+1$$
 then $\boxed{F_V(x)=xF_U(x)}$





$$\begin{split} :&F_V(x) = \sum_{k=0}^\infty V_k x^k = \sum_{k=1}^\infty \underbrace{U_{k-1}} x^k \\ &= x \sum_{j=0}^\infty \underbrace{U_j} x^j = x F_U(x). \end{split}$$



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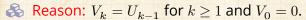




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Generalization of SR2:

(1) If
$$V = U + i$$
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Generalization of SR2:

$$\clubsuit$$
 (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

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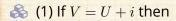






Useful results we'll need for g.f.'s

Generalization of SR2:



$$F_V(x) = x^i F_U(x).$$

 \clubsuit (2) If V = U - i then

$$F_V(x) = x^{-i} F_U(x)$$

$$= x^{-i} \sum_{k=0}^{\infty} U_k x^k$$

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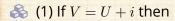
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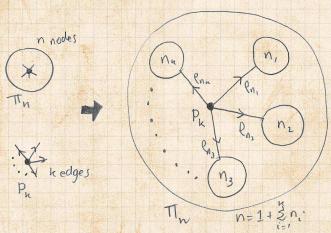
Size of the Giant Component Average Component Size







Goal: figure out forms of the component generating functions, F_{π} and F_{o} .





 $\begin{cases} \& \end{cases}$ Relate π_n to P_k and ρ_n through one step of recursion.

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 $\Re \pi_n$ = probability that a random node belongs to a finite component of size n

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 $\Re \pi_n$ = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array}\right)$$

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Therefore:

$$F_{\pi}(x) =$$



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Therefore:

$$F_{\pi}(x) = \underbrace{F_{P}\left(F_{\rho}(x)\right)}_{\text{SR1}}$$





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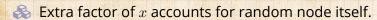
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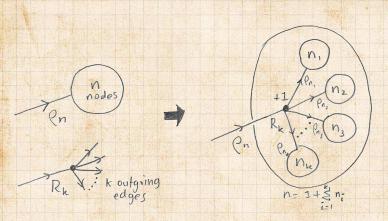
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 \Re Relate ρ_n to R_k and ρ_n through one step of recursion.

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 ρ_n = probability that a random link leads to a finite subcomponent of size n.

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Invoke one step of recursion: ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size n-1,

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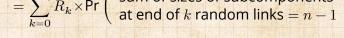
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Connecting generating functions:

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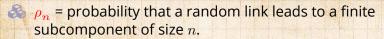
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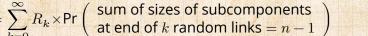
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Therefore:
$$F_{\rho}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_{R}\left(F_{\rho}(x)\right)}_{\text{SR1}}$$

itself.





3



We now have two functional equations connecting our generating functions:

$$F_{\pi}(x) = xF_{P}\left(F_{\rho}(x)\right)$$
 and $F_{\rho}(x) = xF_{R}\left(F_{\rho}(x)\right)$

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 \mathbb{R} Taking stock: We know $F_{P}(x)$ and $F_{R}(x) = F'_{P}(x)/F'_{P}(1).$

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- $\red {\Bbb S}$ We can do this because it only involves $F_
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- $\red {\Bbb S}$ We can do this because it only involves $F_
 ho$ and F_R .
- The first equation then immediately gives us F_{π} in terms of F_{ρ} and F_{R} .



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Remembering vaguely what we are doing:

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Remembering vaguely what we are doing:

Finding F_{π} to obtain the fractional size of the largest component $S_1 = 1 - F_{\pi}(1)$.

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Remembering vaguely what we are doing: Finding F_{π} to obtain the fractional size of the largest component $S_1 = 1 - F_{\pi}(1)$.

Set x = 1 in our two equations:

Solve second education numerically

Plug Full and first equation to dollars Full

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Set x = 1 in our two equations:

$$F_{\pi}(1) = F_{P}\left(F_{\rho}(1)\right) \text{ and } F_{\rho}(1) = F_{R}\left(F_{\rho}(1)\right)$$

Solve second equation numerically

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- $\red {\Bbb S}$ Solve second equation numerically for $F_{
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- $\ensuremath{\mathfrak{S}}$ Plug $F_{\rho}(1)$ into first equation to obtain $F_{\pi}(1)$.

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Example: Standard random graphs.



We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

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Example: Standard random graphs.



We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F_P'(x)/F_P'(1)$$

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 ...aha!

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RHS's of our two equations are the same.

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RHS's of our two equations are the same.

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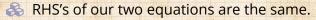
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Consistent with how our dirty (but wrong) trick worked earlier ...

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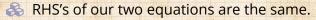
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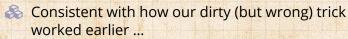
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$$\ensuremath{\mathfrak{S}} \pi_n = \rho_n$$
 just as $P_k = R_k$.

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We are down to

$$F_\pi(x) = x F_R(F_\pi(x))$$
 and $F_R(x) = e^{-\langle k \rangle (1-x)}$.



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$$F_\pi(x) = x F_R(F_\pi(x))$$
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$$: F_{\pi}(x) = xe^{-\langle k \rangle (1 - F_{\pi}(x))}$$

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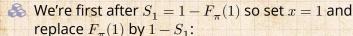


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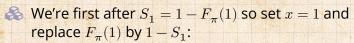


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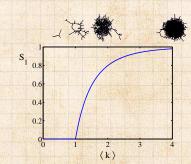


$$:: F_{\pi}(x) = xe^{-\langle k \rangle (1 - F_{\pi}(x))}$$



$$1 - S_1 = e^{-\langle k \rangle S_1}$$

Or:
$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$



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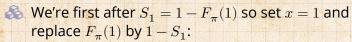


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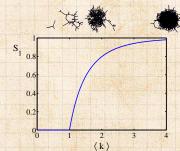


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Just as we found with our dirty trick ...



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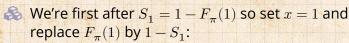


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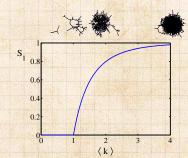


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Just as we found with our dirty trick ...

Again, we (usually) have to resort to numerics ...

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Notation: The Kronicker delta function Ω $\delta_{ij}=1$ if i=j and 0 otherwise.

$$P_k = \delta_{k1}$$

$$P_k = \delta_{k2}$$

$$P_k = \delta_k$$

 $P_k = \delta_{kk'}$ for some fixed $k' \ge 0$.

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$$

$$P_k = a\delta_{k1} + (1-a)\delta_{k3}$$
, with $0 \le a \le a$

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{kk'}$$
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if i = j and 0 otherwise.

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 \Re Notation: The Kronecker delta function $\Im \delta_{ij} = 1$ if i = j and 0 otherwise.



$$P_k = \delta_{k1}.$$

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Notation: The Kronecker delta function $\delta_{ij} = 1$ if i = j and 0 otherwise.

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 $P_k = \delta_{kk'}$ for some fixed $k' \ge 0$.

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$$

 $P_k = a\delta_{k,1} + (1-a)\delta_{k,3}$, with $0 \le a \le 1$

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$$\begin{split} P_k &= a\delta_{k1} + (1-a)\delta_{k3} \text{, with } 0 \leq a \leq 1. \\ P_k &= \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{kk'} \text{ for some fixed } k' \geq 2. \\ P_k &= a\delta_{k1} + (1-a)\delta_{kk'} \text{ for some fixed } k' \geq 2 \text{ with } 0 \leq a \leq 1. \end{split}$$

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 $P_k = a\delta_{k\,k} + (1-a)\delta_{k\,k'}$ for some fixed $k' \geq 2$ with $0 \leq a \leq 1$

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A joyful example □:

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$



Arr We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.

$$F_P(x) = rac{1}{2}x + rac{1}{2}x^3$$
 and $F_R(x) = rac{1}{4}x^0 + rac{3}{4}x^2$

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A giant component exists because:

$$\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1.$$

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- & Generating functions for P_k and R_k :

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

Check for goodness:

Things to figure out: Component size generating functions for π_n and ρ_n , and the size of the giant component

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Check for goodness:

 $F_R(x)=F_P'(x)/F_P'(1)$ and $F_P(1)=F_R(1)$ $F_P'(1)=\langle k \rangle_P=2$ and $F_R'(1)=\langle k \rangle_R=rac{3}{2}$.

Things to figure out: Component size generating functions for π_n and ρ_n , and the size of the giant component

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Things to figure out: Component size generating functions for π_n and ρ_n , and the size of the giant component



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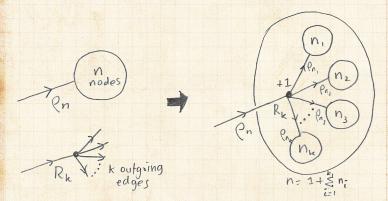


Find $F_{\rho}(x)$ first:



We know:

$$F_{\rho}(x) = xF_{R}(F_{\rho}(x)).$$



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$$F_{\rho}(x) = x \left(\frac{1}{4} + \frac{3}{4} \left[F_{\rho}(x)\right]^2\right).$$

$$3x \left[F_{\rho}(x)\right]^2 - 4F_{\rho}(x) + x = 0$$

$$F_{\rho}(x) = \frac{2}{3x} \left(1 \pm \sqrt{1 + \frac{3}{4}x^2} \right)$$

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$$F_{\rho}(x) = x \left(\frac{1}{4} + \frac{3}{4} \left[F_{\rho}(x) \right]^2 \right).$$

Rearranging:

$$3x \left[F_{\rho}(x) \right]^2 - 4F_{\rho}(x) + x = 0.$$

Please and thank you:

$$F_{\rho}(x) = \frac{2}{3x} \left(1 \pm \sqrt{1 - \frac{3}{4}x^2} \right)$$

Time for a Taylor series expansion.

The promise: non-negative powers of x with non-negative coefficients.

First: which sign do we take?

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$$(1+z)^{\theta} = {\theta \choose 0} z^0 + {\theta \choose 1} z^1 + {\theta \choose 2} z^2 + {\theta \choose 2} z^3 + \dots$$

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3 Thinking about the limit $x \to 0$ in

$$F_{\rho}(x) = \frac{2}{3x} \left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right), \label{eq:free_point}$$

we see that the positive sign solution blows to smithereens, and the negative one is okay.

$$F_{\rho}(x) = \frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right)$$

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we see that the positive sign solution blows to smithereens, and the negative one is okay.



So we must have:

$$F_{\rho}(x) = \frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right),$$



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So we must have:

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We can now deploy the Taylor expansion:

$$(1+z)^{\theta} = {\theta \choose 0} z^0 + {\theta \choose 1} z^1 + {\theta \choose 2} z^2 + {\theta \choose 2} z^3 + \dots$$



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Let's define a binomial for arbitrary θ and k = 0, 1, 2, ...:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

$$(1+z)^{rac{1}{2}}={rac{1}{2}\choose 0}z^0+{rac{1}{2}\choose 1}z^1+{rac{1}{2}\choose 2}z^2+.$$

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$$=1+\frac{1}{2}z-\frac{1}{8}z^2+\frac{1}{16}z^3$$



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$$(1+z)^{\frac{1}{2}} = {\frac{1}{2} \choose 0} z^0 + {\frac{1}{2} \choose 1} z^1 + {\frac{1}{2} \choose 2} z^2 + \dots$$

$$= \frac{\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(\frac{3}{2})}z^0 + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})}z^1 + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)\Gamma(-\frac{1}{2})}z^2 + \dots$$



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where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.



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where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.



Solution Note: $(1+z)^{\theta} \sim 1 + \theta z$ always.



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$$F_{\rho}(x)=\frac{2}{3x}\left(1-\sqrt{1-\frac{3}{4}x^2}\right).$$

$$F_{\rho}(x) =$$

$$\frac{2}{3x} \left(1 - \left[1 + \frac{1}{2} \left(-\frac{3}{4} x^2 \right)^1 - \frac{1}{8} \left(-\frac{3}{4} x^2 \right)^2 + \frac{1}{16} \left(-\frac{3}{4} x^2 \right) \right]$$

$$F_{
ho}(x) = \sum_{n=0}^{\infty}
ho_n x^n$$

$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \ldots + \frac{2}{3}\left(\frac{3}{4}\right)^k \frac{(-1)^{k+1}\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)}x^{2k-1}$$

$$F_{\rho}(x)=\frac{2}{3x}\left(1-\sqrt{1-\frac{3}{4}x^2}\right).$$

3 Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$F_{\rho}(x) =$$

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🚳 Giving:

$$F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \ldots + \frac{2}{3}\left(\frac{3}{4}\right)^k \frac{(-1)^{k+1}\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)}x^{2k-1} + \ldots$$

Do odd powers make sense?

$$F_{\rho}(x)=\frac{2}{3x}\left(1-\sqrt{1-\frac{3}{4}x^2}\right).$$

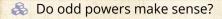
Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$F_{\rho}(x) =$$

$$\frac{2}{3x} \left(1 - \left[1 + \frac{1}{2} \left(-\frac{3}{4} x^2 \right)^1 - \frac{1}{8} \left(-\frac{3}{4} x^2 \right)^2 + \frac{1}{16} \left(-\frac{3}{4} x^2 \right)^3 \right] + \dots \right)$$

$$F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \ldots + \frac{2}{3}\left(\frac{3}{4}\right)^k \frac{(-1)^{k+1}\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)}x^{2k-1} + \ldots$$





$$F_{\pi}(x) = x F_P\left(F_{\pi}(x)\right)$$

$$= x \frac{1}{2} \left(\left(F_{\rho}(x) \right)^{1} + \left(F_{\rho}(x) \right)^{3} \right)$$

$$= x \frac{1}{2} \left[\frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^{2}} \right) + \frac{2^{3}}{(3x)^{3}} \left(1 - \sqrt{1 - \frac{3}{4}x^{2}} \right) \right]$$

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$$F_{\pi}(x) = x F_P\left(F_{\pi}(x)\right)$$

$$=x\frac{1}{2}\left(\left(F_{\rho}(x)\right)^{1}+\left(F_{\rho}(x)\right)^{3}\right)$$

$$= x \frac{1}{2} \left[\frac{2}{3x} \left(1 + \sqrt{1 - \frac{3}{4}x^2} \right) + \frac{2^3}{(3x)^3} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right) \right]$$

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$$F_{\pi}(x) = xF_{P}\left(F_{\pi}(x)\right)$$

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Delicious.



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$$=x\frac{1}{2}\left(\left(F_{\rho}(x)\right)^{1}+\left(F_{\rho}(x)\right)^{3}\right)$$

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Delicious.



In principle, we can now extract all the π_n .



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 $\begin{cases} \& \& \end{cases}$ We can now find $F_{\pi}(x)$ with:

$$F_{\pi}(x) = x F_P\left(F_{\pi}(x)\right)$$

$$=x\frac{1}{2}\left(\left(F_{\rho}(x)\right)^{1}+\left(F_{\rho}(x)\right)^{3}\right)$$

$$= x \frac{1}{2} \left[\frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right) + \frac{2^3}{(3x)^3} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right)^3 \right].$$





Delicious.



In principle, we can now extract all the π_n .



But let's just find the size of the giant component.



$$\left. F_{\rho}(x) \right|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4} 1^2} \right) = \frac{1}{3}.$$

$$F_{\pi}(1) = 1 \cdot F_{P}\left(F_{\rho}(1)\right)$$

$$S_1 = 1 - F_{\pi}(1) = 1 - \frac{5}{27} = \frac{22}{27}$$

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$$\left. F_{\rho}(x) \right|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4} 1^2} \right) = \frac{1}{3}.$$

This is the probability that a random edge leads to a sub-component of finite size.

$$S_1 = 1 - F_{\pi}(1) = 1 - \frac{5}{27} = \frac{22}{27}$$

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$$\left. F_{\rho}(x) \right|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4} 1^2} \right) = \frac{1}{3}.$$

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- Next:

$$F_{\pi}(1) = 1 \cdot F_{P}\left(F_{\rho}(1)\right) = F_{P}\left(\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \left(\frac{1}{3}\right)^{3} = \frac{5}{27}$$

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$$F_{\pi}(1) = 1 \cdot F_{P} \left(F_{\rho}(1) \right) \\ = F_{P} \left(\frac{1}{3} \right) \\ = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \left(\frac{1}{3} \right)^{3} \\ = \frac{5}{27}.$$

$$S_1 = 1 - F_{\pi}(1) = 1 - \frac{5}{27} = \frac{22}{27}$$

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$$\left. F_{\rho}(x) \right|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4} 1^2} \right) = \frac{1}{3}.$$

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$$F_{\pi}(1) = 1 \cdot F_{P}\left(F_{\rho}(1)\right) = F_{P}\left(\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right)^{3} = \frac{5}{27}.$$

- This is the probability that a random chosen node belongs to a finite component.

$$1 - F_{\pi}(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$

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$$\left. F_{\rho}(x) \right|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4} 1^2} \right) = \frac{1}{3}.$$

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- This is the probability that a random chosen node belongs to a finite component.
- Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$



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Next: find average size of finite components $\langle n \rangle$.

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 \mathbb{A} Next: find average size of finite components $\langle n \rangle$.

 \mathbb{R} Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.

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Next: find average size of finite components $\langle n \rangle$.

 $\ref{Mathematics}$ Using standard G.F. result: $\langle n \rangle = F_\pi'(1)$.

 $\red {
m S}$ Try to avoid finding $F_{\pi}(x)$...

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- \mathbb{A} Next: find average size of finite components $\langle n \rangle$.
- \Leftrightarrow Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.
- $\red{\red{\red{S}}}$ Try to avoid finding $F_{\pi}(x)$...
- Starting from $F_{\pi}(x) = xF_{P}\left(F_{\rho}(x)\right)$, we differentiate:

$$F_{\pi}'(x) = F_{P}\left(F_{\rho}(x)\right) + xF_{\rho}'(x)F_{P}'\left(F_{\rho}(x)\right)$$

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$$F_{\pi}'(x) = F_{P}\left(F_{\rho}(x)\right) + xF_{\rho}'(x)F_{P}'\left(F_{\rho}(x)\right)$$

 \Longrightarrow While $F_{\rho}(x)=xF_{R}\left(F_{\rho}(x)\right)$ gives

$$F_{\rho}'(x) = F_{R}\left(F_{\rho}(x)\right) + xF_{\rho}'(x)F_{R}'\left(F_{\rho}(x)\right)$$

N. T.



- Next: find average size of finite components $\langle n \rangle$.
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Now set x = 1 in both equations.

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- Now set x = 1 in both equations.
- We solve the second equation for $F'_{\rho}(1)$ (we must already have $F_{\rho}(1)$).

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- Now set x = 1 in both equations.
- We solve the second equation for $F'_{\rho}(1)$ (we must already have $F_{\rho}(1)$).
- Plug $F'_{\rho}(1)$ and $F_{\rho}(1)$ into first equation to find $F'_{\pi}(1)$.

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Example: Standard random graphs.

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Example: Standard random graphs.



 \blacksquare Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.

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Example: Standard random graphs.



Use fact that $F_P = F_R$ and $F_\pi = F_o$.



Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_P(F_{\pi}(x)) + xF'_{\pi}(x)F'_P(F_{\pi}(x))$$

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Example: Standard random graphs.



Use fact that $F_P = F_R$ and $F_\pi = F_o$.



Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_P(F_{\pi}(x)) + xF'_{\pi}(x)F'_P(F_{\pi}(x))$$

Rearrange:
$$F'_{\pi}(x) = \frac{F_{P}(F_{\pi}(x))}{1 - xF'_{P}(F_{\pi}(x))}$$

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Example: Standard random graphs.



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Rearrange:
$$F'_{\pi}(x) = \frac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$



 \Longrightarrow Simplify denominator using $F_P(x) = \langle k \rangle F_P(x)$

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Example: Standard random graphs.



Use fact that $F_P = F_R$ and $F_\pi = F_o$.



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 \Longrightarrow Simplify denominator using $F_P(x) = \langle k \rangle F_P(x)$



Replace $F_P(F_{\pi}(x))$ using $F_{\pi}(x) = xF_P(F_{\pi}(x))$.

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Example: Standard random graphs.

- $\red {\Bbb S}$ Use fact that $F_P=F_R$ and $F_\pi=F_
 ho.$
- Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_P(F_{\pi}(x)) + xF'_{\pi}(x)F'_P(F_{\pi}(x))$$

Rearrange:
$$F_{\pi}'(x) = \frac{F_P(F_{\pi}(x))}{1 - xF_P'(F_{\pi}(x))}$$

- \red{shift} Simplify denominator using $F_P'(x) = \langle k \rangle F_P(x)$
- Set x = 1 and replace $F_{\pi}(1)$ with $1 S_1$.

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Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_o$.
- Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_P(F_{\pi}(x)) + xF'_{\pi}(x)F'_P(F_{\pi}(x))$$

Rearrange:
$$F_{\pi}'(x) = \frac{F_P\left(F_{\pi}(x)\right)}{1 - xF_P'\left(F_{\pi}(x)\right)}$$



 \Longrightarrow Simplify denominator using $F_P(x) = \langle k \rangle F_P(x)$



Replace $F_{\mathcal{P}}(F_{\pi}(x))$ using $F_{\pi}(x) = xF_{\mathcal{P}}(F_{\pi}(x))$.

 \Longrightarrow Set x=1 and replace $F_{\pi}(1)$ with $1-S_1$.

End result:
$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$



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Our result for standard random networks:

$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$

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Our result for standard random networks:

$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$

Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.

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Our result for standard random networks:

$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.

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Our result for standard random networks:

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- \clubsuit We have $S_1 = 0$ for all $\langle k \rangle < 1$

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Our result for standard random networks:

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- $\red {\$}$ We have $S_1=0$ for all $\langle k \rangle <1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

 \clubsuit This blows up as $\langle k \rangle \to 1$.

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- \clubsuit This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.

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Our result for standard random networks:

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- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- $\red {\$}$ We have $S_1=0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- \clubsuit This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.
- Typical critical point behavior ...

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 \Longrightarrow Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$

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 \Longrightarrow Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$



 $As \langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.

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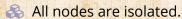


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$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$



 \Leftrightarrow As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.



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 \Longrightarrow Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$



 $As \langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.



All nodes are isolated.



 $As \langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.

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 \Longrightarrow Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F_\pi'(1) = \frac{(1-S_1)}{1-\langle k \rangle (1-S_1)}$$

- $As \langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.
- All nodes are isolated.
- $As \langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

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Extra on largest component size:

 \Longrightarrow For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.

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- All nodes are isolated.
- $As \langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

Extra on largest component size:

- \Leftrightarrow For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.
- \Longrightarrow For $\langle k \rangle < 1$, $S_1 \sim (\log N)/N$.

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$$\langle n \rangle = F_\pi'(1) = F_P\left(F_\rho(1)\right) + F_\rho'(1)F_P'\left(F_\rho(1)\right)$$

$$F_P(x)=\frac{1}{2}x+\frac{1}{2}x^3 \text{ and } F_R(x)=\frac{1}{4}x^0+\frac{3}{4}x^2$$

$$F_P'(x) = rac{1}{2} + rac{3}{2} x^2 ext{ and } F_R'(x) = rac{3}{2} x^2$$



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We're after:

$$\langle n \rangle = F_\pi'(1) = F_P\left(F_\rho(1)\right) + F_\rho'(1)F_P'\left(F_\rho(1)\right)$$

$$F_{\rho}'(1) = F_{R}(F_{\rho}(1)) + F_{\rho}'(1)F_{R}'(F_{\rho}(1))$$

$$F_P(x)=\frac{1}{2}x+\frac{1}{2}x^3 \text{ and } F_R(x)=\frac{1}{4}x^0+\frac{3}{4}x^2$$

$$F_P'(x) = rac{1}{2} + rac{3}{2}x^2$$
 and $F_B'(x) = rac{3}{2}x^2$



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We're after:

$$\langle n \rangle = F_\pi'(1) = F_P\left(F_\rho(1)\right) + F_\rho'(1)F_P'\left(F_\rho(1)\right)$$

where we first need to compute

$$F_{\rho}'(1) = F_R \left(F_{\rho}(1)\right) + F_{\rho}'(1) F_R' \left(F_{\rho}(1)\right). \label{eq:free_point}$$

$$F_P(x)=\frac{1}{2}x+\frac{1}{2}x^3$$
 and $F_R(x)=\frac{1}{4}x^0+\frac{3}{4}x^2$

$$F_P'(x) = \frac{1}{2} + \frac{3}{2}x^2$$
 and $F_B'(x) = \frac{3}{2}x$



Generating

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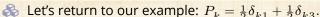
Useful results Component

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We're after:

$$\langle n \rangle = F_\pi'(1) = F_P\left(F_\rho(1)\right) + F_\rho'(1)F_P'\left(F_\rho(1)\right)$$

where we first need to compute

$$F_{\rho}'(1) = F_R \left(F_{\rho}(1) \right) + F_{\rho}'(1) F_R' \left(F_{\rho}(1) \right). \label{eq:free_point}$$



Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$

$$F_P'(x) = \frac{1}{2} + \frac{3}{2}x^2$$
 and $F_R'(x) = \frac{3}{2}x^2$



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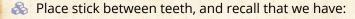


We're after:

$$\langle n \rangle = F_\pi'(1) = F_P\left(F_\rho(1)\right) + F_\rho'(1)F_P'\left(F_\rho(1)\right)$$

where we first need to compute

$$F_{\rho}'(1) = F_R \left(F_{\rho}(1)\right) + F_{\rho}'(1) F_R' \left(F_{\rho}(1)\right). \label{eq:free_point}$$



$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$

Differentiation gives us:

$$F_P'(x) = \frac{1}{2} + \frac{3}{2} x^2 \text{ and } F_R'(x) = \frac{3}{2} x.$$



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$$F_{\rho}'(1) = F_R\left(F_{\rho}(1)\right) + F_{\rho}'(1)F_R'\left(F_{\rho}(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_{\rho}(1)F'_R\left(\frac{1}{3}\right)$$
$$= \frac{1}{4} + \frac{3}{4}\frac{1}{3^2} + F'_{\rho}(1)\frac{3}{2}\frac{1}{3}.$$

Finally:
$$\langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right)$$

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$$F'_{\rho}(1) = F_{R}\left(F_{\rho}(1)\right) + F'_{\rho}(1)F'_{R}\left(F_{\rho}(1)\right)$$

$$= F_{R}\left(\frac{1}{3}\right) + F'_{\rho}(1)F'_{R}\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{3}{4}\frac{1}{23} + F'_{\rho}(1)\frac{3}{2}\frac{1}{2}$$

Finally:
$$\langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right)$$

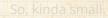
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$$\begin{split} F_{\rho}'(1) &= F_{R} \left(F_{\rho}(1) \right) + F_{\rho}'(1) F_{R}' \left(F_{\rho}(1) \right) \\ &= F_{R} \left(\frac{1}{3} \right) + F_{\rho}'(1) F_{R}' \left(\frac{1}{3} \right) \\ &= \frac{1}{4} + \frac{\cancel{3}}{4} \frac{1}{\cancel{3}^{\cancel{2}}} + F_{\rho}'(1) \frac{\cancel{3}}{2} \frac{1}{\cancel{3}}. \end{split}$$

Finally:
$$\langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right)$$

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$$\begin{split} F_{\rho}'(1) &= F_{R}\left(F_{\rho}(1)\right) + F_{\rho}'(1)F_{R}'\left(F_{\rho}(1)\right) \\ &= F_{R}\left(\frac{1}{3}\right) + F_{\rho}'(1)F_{R}'\left(\frac{1}{3}\right) \\ &= \frac{1}{4} + \frac{3}{4}\frac{1}{3^{2}} + F_{\rho}'(1)\frac{3}{2}\frac{1}{3}. \end{split}$$

After some reallocation of objects, we have $F_0'(1) = \frac{13}{2}$.

Finally:
$$\langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right)$$



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$$\begin{split} F_{\rho}'(1) &= F_R\left(F_{\rho}(1)\right) + F_{\rho}'(1)F_R'\left(F_{\rho}(1)\right) \\ \\ &= F_R\left(\frac{1}{3}\right) + F_{\rho}'(1)F_R'\left(\frac{1}{3}\right) \\ \\ &= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{\cancel{3}\cancel{2}} + F_{\rho}'(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}. \end{split}$$

After some reallocation of objects, we have $F_a'(1) = \frac{13}{3}$.



Finally:
$$\langle n \rangle = F_{\pi}'(1) = F_{P}\left(\frac{1}{3}\right) + \frac{13}{2}F_{P}'\left(\frac{1}{3}\right)$$

$$\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^{3}} + \frac{13}{2}\left(\frac{1}{2} + \frac{3}{2}\frac{1}{3^{2}}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27}$$



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$$\begin{split} F_{\rho}'(1) &= F_R \left(F_{\rho}(1) \right) + F_{\rho}'(1) F_R' \left(F_{\rho}(1) \right) \\ \\ &= F_R \left(\frac{1}{3} \right) + F_{\rho}'(1) F_R' \left(\frac{1}{3} \right) \\ \\ &= \frac{1}{4} + \frac{3}{4} \frac{1}{32} + F_{\rho}'(1) \frac{3}{2} \frac{1}{3}. \end{split}$$

After some reallocation of objects, we have $F_0'(1) = \frac{13}{2}$.



$$\begin{split} & \text{Finally: } \langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right) \\ &= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{3\cancel{2}}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27} \end{split}$$



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$$\begin{split} F_\rho'(1) &= F_R \left(F_\rho(1) \right) + F_\rho'(1) F_R' \left(F_\rho(1) \right) \\ \\ &= F_R \left(\frac{1}{3} \right) + F_\rho'(1) F_R' \left(\frac{1}{3} \right) \\ \\ &= \frac{1}{4} + \frac{3}{4} \frac{1}{3^2} + F_\rho'(1) \frac{3}{2} \frac{1}{3}. \end{split}$$

After some reallocation of objects, we have $F'_{o}(1) = \frac{13}{2}$.



$$\begin{split} & \text{Finally: } \langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right) \\ & = \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{3}{2}\frac{1}{3^2}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27} \end{split}$$

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$$\begin{split} F_{\rho}'(1) &= F_R\left(F_{\rho}(1)\right) + F_{\rho}'(1)F_R'\left(F_{\rho}(1)\right) \\ \\ &= F_R\left(\frac{1}{3}\right) + F_{\rho}'(1)F_R'\left(\frac{1}{3}\right) \\ \\ &= \frac{1}{4} + \frac{3}{4}\frac{1}{2^2} + F_{\rho}'(1)\frac{3}{2}\frac{1}{3}. \end{split}$$

After some reallocation of objects, we have $F'_{o}(1) = \frac{13}{2}$.



$$\begin{split} & \text{Finally: } \langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right) \\ &= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{\cancel{3}\cancel{2}}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27} \,. \end{split}$$



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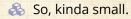


$$\begin{split} F_{\rho}'(1) &= F_R \left(F_{\rho}(1) \right) + F_{\rho}'(1) F_R' \left(F_{\rho}(1) \right) \\ \\ &= F_R \left(\frac{1}{3} \right) + F_{\rho}'(1) F_R' \left(\frac{1}{3} \right) \\ \\ &= \frac{1}{4} + \frac{3}{4} \frac{1}{3^2} + F_{\rho}'(1) \frac{3}{2} \frac{1}{3}. \end{split}$$

After some reallocation of objects, we have $F_0'(1) = \frac{13}{2}$.



$$\begin{split} & \text{Finally: } \langle n \rangle = F_\pi'(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F_P'\left(\frac{1}{3}\right) \\ &= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{\cancel{3}^2}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27}. \end{split}$$





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Generating functions allow us to strangely calculate features of random networks.

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- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.

We'll find generating functions useful for contagion.

But we'll also see that more direct, physics-bearing calculations are possible.

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Elevation:

COcoNuTS -

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