Generating Functions and Networks

Complex Networks | @networksvox
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COcoNuTS
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
Average Component Size
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Outline

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Generatingfunctionology [1]

- Idea: Given a sequence $a_0, a_1, a_2, \ldots$, associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Roughly: transforms a vector in R^∞ into a function defined on R^1.
- Related to Fourier, Laplace, Mellin, ...

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Simple examples:

Rolling dice and flipping coins:

- $p_k^{(⚁)} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

$$F^{(⚁)}(x) = \sum_{k=1}^{6} p_k^{(⚁)} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

- $p_0^{(\text{coin})} = \mathbf{Pr}(\text{head}) = 1/2$, $p_1^{(\text{coin})} = \mathbf{Pr}(\text{tail}) = 1/2$.

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})} x^0 + p_1^{(\text{coin})} x^1 = \frac{1}{2}(1 + x).$$

- A generating function for a probability distribution is called a Probability Generating Function (p.g.f.).
- We'll come back to these simple examples as we derive various delicious properties of generating functions.

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Example

- Take a degree distribution with exponential decay:

$$P_k = c e^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} c e^{-\lambda k} x^k = \frac{c}{1 - x e^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.
- For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1.$$

- Check die and coin p.g.f.'s.

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Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \bigg|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda}) e^{-\lambda}}{(1 - x e^{-\lambda})^2}.$$

-

So: $\langle k \rangle = F'(1) = \dfrac{e^{-\lambda}}{(1 - e^{-\lambda})}$.

- Check for die and coin p.g.f.'s.

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Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

- First moment:

$$\langle k \rangle = F'(1)$$

- Higher moments:

$$\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \bigg|_{x=1}$$

- kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \bigg|_{x=0}$$

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A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x) F_V(x).$$

- Convolve yourself with Convolutions: Insert question from assignment 5 .
- Try with die and coin p.g.f.'s.
  1. Add two coins (tail=0, head=1).
  2. Add two dice.
  3. Add a coin flip to one die roll.

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Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's re-express our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:
  $F_P(x)$ is the g.f. for P_k.
  $F_R(x)$ is the g.f. for R_k.
- Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F_R'(1) > 1.$$

- Now find how F_R is related to F_P ...

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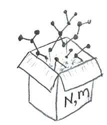

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Edge-degree distribution

- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1) P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0 \right) = \frac{1}{\langle k \rangle} F_P'(x).$$

Finally, since $\langle k \rangle = F_P'(1)$,

$$\boxed{F_R(x) = \frac{F_P'(x)}{F_P'(1)}}$$

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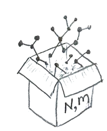


Edge-degree distribution

- Recall giant component condition is $\langle k \rangle_R = F_R'(1) > 1$.
- Since we have $F_R(x) = F_P'(x)/F_P'(1)$,

$$F_R'(x) = \frac{F_P''(x)}{F_P'(1)}.$$

- Setting $x = 1$, our condition becomes

$$\boxed{\frac{F_P''(1)}{F_P'(1)} > 1}$$

Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Definitions:

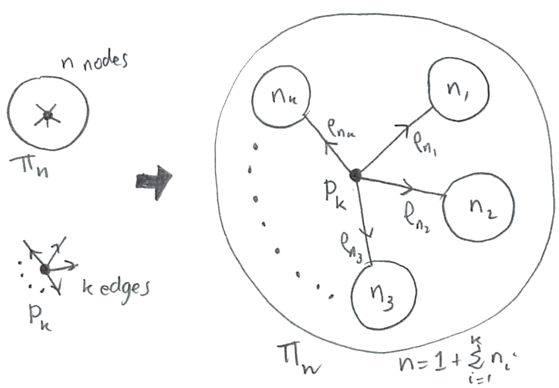
- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
- ρ_n = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

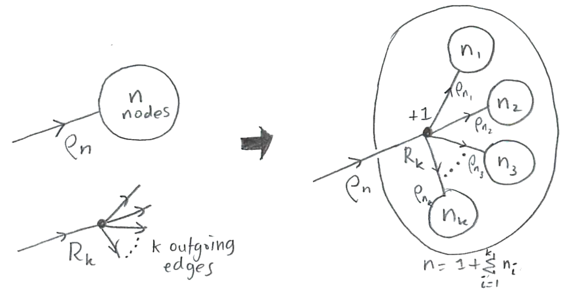
neighbors $\Leftrightarrow$ components

Connecting probabilities:



- Markov property of random networks connects π_n, ρ_n, and P_k.

Connecting probabilities:



- Markov property of random networks connects ρ_n and R_k.

G.f.'s for component size distributions:

-
$$F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

- Subtle key: $F_\pi(1)$ is the probability that a node belongs to a finite component.
- Therefore: $S_1 = 1 - F_\pi(1)$.

Our mission, which we accept:

- Determine and connect the four generating functions

$$F_P, F_R, F_\pi, \text{ and } F_\rho.$$

Useful results we'll need for g.f.'s

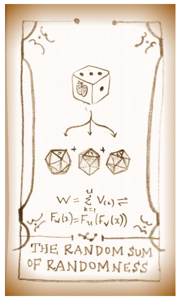
Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{U} V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

then

$$\boxed{F_W(x) = F_U\left(F_V(x)\right)}$$



Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \ldots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x).$$

Proof of SR1:

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\}| \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\}| \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j} x^k$$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \ldots, i_j\}| \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}$$

Useful results we'll need for g.f.'s

Generalization of SR2:

- (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

- (2) If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x)$$

$$= x^{-i} \sum_{k=0}^{\infty} U_k x^k$$

Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \underbrace{\underbrace{\sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \ldots, i_j\}| \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j = (F_V(x))^j}$$

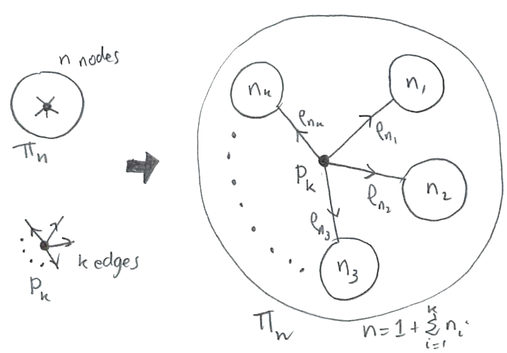
$$= \sum_{j=0}^{\infty} U_j \left(F_V(x)\right)^j$$

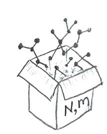
$$= F_U\left(F_V(x)\right)$$

- Alternate, groovier proof in the accompanying assignment.

Connecting generating functions:

- Goal: figure out forms of the component generating functions, F_π and F_ρ.





- Relate π_n to P_k and ρ_n through one step of recursion.

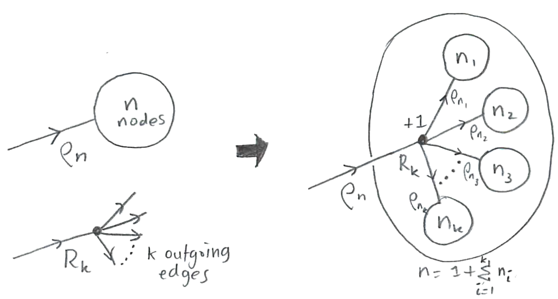
Connecting generating functions:

- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{l}\text{sum of sizes of subcomponents}\\ \text{at end of } k \text{ random links} = n-1\end{array}\right)$$

- Therefore: $\boxed{F_\pi(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_P\left(F_\rho(x)\right)}_{\text{SR1}}}$
- Extra factor of x accounts for random node itself.

Connecting generating functions:



- Relate ρ_n to R_k and ρ_n through one step of recursion.

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Connecting generating functions:

- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion:
  ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n-1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr\left(\begin{array}{l}\text{sum of sizes of subcomponents}\\ \text{at end of } k \text{ random links} = n-1\end{array}\right)$$

- Therefore: $\boxed{F_\rho(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_R\left(F_\rho(x)\right)}_{\text{SR1}}}$
- Again, extra factor of x accounts for random node itself.

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Connecting generating functions:

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F_P'(x)/F_P'(1)$.
- We first untangle the second equation to find F_ρ
- We can do this because it only involves F_ρ and F_R.
- The first equation then immediately gives us F_π in terms of F_ρ and F_R.

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- Remembering vaguely what we are doing:
  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.
- Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P\left(F_\rho(1)\right) \text{ and } F_\rho(1) = F_R\left(F_\rho(1)\right)$$

- Solve second equation numerically for $F_\rho(1)$.
- Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$.

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Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F_P'(x)/F_P'(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x) \qquad \text{...aha!}$$

- RHS's of our two equations are the same.
- So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$
- Consistent with how our dirty (but wrong) trick worked earlier ...
- $\pi_n = \rho_n$ just as $P_k = R_k$.

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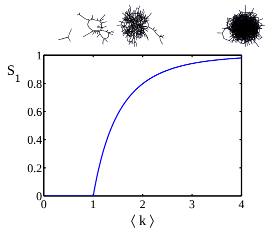
- We are down to
  $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k \rangle (1-x)}$.
- $$\therefore F_\pi(x) = x e^{-\langle k \rangle (1 - F_\pi(x))}$$
- We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

Or: $$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$



- Just as we found with our dirty trick ...
- Again, we (usually) have to resort to numerics ...

A few simple random networks to contemplate and play around with:

- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.
- $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- $P_k = a\delta_{k1} + (1-a)\delta_{k3}$, with $0 \leq a \leq 1$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{kk'}$ for some fixed $k' \geq 2$.
- $P_k = a\delta_{k1} + (1-a)\delta_{kk'}$ for some fixed $k' \geq 2$ with $0 \leq a \leq 1$.

A joyful example:

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1$.
- Generating functions for P_k and R_k:

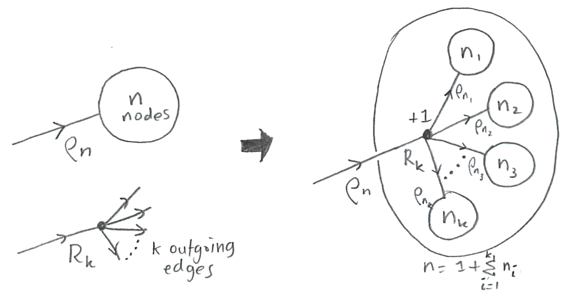
$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

- Check for goodness:
  - $F_R(x) = F'_P(x)/F'_P(1)$ and $F_P(1) = F_R(1) = 1$.
  - $F'_P(1) = \langle k \rangle_P = 2$ and $F'_R(1) = \langle k \rangle_R = \frac{3}{2}$.
- Things to figure out: Component size generating functions for π_n and ρ_n, and the size of the giant component.

Find $F_\rho(x)$ first:

- We know:

$$F_\rho(x) = xF_R\left(F_\rho(x)\right).$$



- Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

- Rearranging:

$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$

- Please and thank you:

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$

- Time for a Taylor series expansion.
- The promise: non-negative powers of x with non-negative coefficients.
- First: which sign do we take?

- Because ρ_n is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.
- Thinking about the limit $x \to 0$ in

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right),$$

we see that the positive sign solution blows to smithereens, and the negative one is okay.
- So we must have:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right),$$

- We can now deploy the Taylor expansion:

$$(1+z)^\theta = \binom{\theta}{0}z^0 + \binom{\theta}{1}z^1 + \binom{\theta}{2}z^2 + \binom{\theta}{2}z^3 + \ldots$$

- Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \dots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

- For $\theta = \frac{1}{2}$, we have:

$$(1+z)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \dots$$

$$= \frac{\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(\frac{3}{2})} z^0 + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})} z^1 + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)\Gamma(-\frac{1}{2})} z^2 + \dots$$

$$= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \dots$$

where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

- Note: $(1+z)^\theta \sim 1 + \theta z$ always.

- Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right).$$

- Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$F_\rho(x) =$$

$$\frac{2}{3x}\left(1 - \left[1 + \frac{1}{2}\left(-\frac{3}{4}x^2\right)^1 - \frac{1}{8}\left(-\frac{3}{4}x^2\right)^2 + \frac{1}{16}\left(-\frac{3}{4}x^2\right)^3\right] + \dots\right)$$

- Giving:

$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \dots + \frac{2}{3}\left(\frac{3}{4}\right)^k \frac{(-1)^{k+1}\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)} x^{2k-1} + \dots$$

- Do odd powers make sense?

- We can now find $F_\pi(x)$ with:

$$F_\pi(x) = xF_P\left(F_\pi(x)\right)$$

$$= x\frac{1}{2}\left(\left(F_\rho(x)\right)^1 + \left(F_\rho(x)\right)^3\right)$$

$$= x\frac{1}{2}\left[\frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right) + \frac{2^3}{(3x)^3}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right)^3\right].$$

- Delicious.
- In principle, we can now extract all the π_n.
- But let's just find the size of the giant component.

- First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3 \cdot 1}\left(1 - \sqrt{1 - \frac{3}{4}1^2}\right) = \frac{1}{3}.$$

- This is the probability that a random edge leads to a sub-component of finite size.
- Next:

$$F_\pi(1) = 1 \cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right)^3 = \frac{5}{27}.$$

- This is the probability that a random chosen node belongs to a finite component.
- Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$

Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- Try to avoid finding $F_\pi(x)$ ...
- Starting from $F_\pi(x) = xF_P\left(F_\rho(x)\right)$, we differentiate:

$$F'_\pi(x) = F_P\left(F_\rho(x)\right) + xF'_\rho(x)F'_P\left(F_\rho(x)\right)$$

- While $F_\rho(x) = xF_R\left(F_\rho(x)\right)$ gives

$$F'_\rho(x) = F_R\left(F_\rho(x)\right) + xF'_\rho(x)F'_R\left(F_\rho(x)\right)$$

- Now set $x = 1$ in both equations.
- We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).
- Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.

Average component size

Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P\left(F_\pi(x)\right) + xF'_\pi(x)F'_P\left(F_\pi(x)\right)$$

$$\text{Rearrange:} \quad F'_\pi(x) = \frac{F_P\left(F_\pi(x)\right)}{1 - xF'_P\left(F_\pi(x)\right)}$$

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
- Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.
- Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

$$\text{End result:} \quad \langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

Average component size

- Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.
- Typical critical point behavior ...

Average component size

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

Extra on largest component size:

- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.
- For $\langle k \rangle < 1$, $S_1 \sim (\log N)/N$.

- Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- We're after:

$$\langle n \rangle = F'_\pi(1) = F_P\left(F_\rho(1)\right) + F'_\rho(1) F'_P\left(F_\rho(1)\right)$$

where we first need to compute

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right).$$

- Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$

- Differentiation gives us:

$$F'_P(x) = \frac{1}{2} + \frac{3}{2}x^2 \text{ and } F'_R(x) = \frac{3}{2}x.$$

- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3\cancel{^2}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

- After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.
- Finally: $\langle n \rangle = F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F'_P\left(\frac{1}{3}\right)$

$$= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{3\cancel{^2}}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27}.$$

- So, kinda small.

Nutshell

- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.
- We'll find generating functions useful for contagion.
- But we'll also see that more direct, physics-bearing calculations are possible.

References I

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