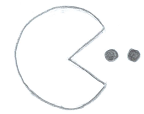
MATH 124: Matrixology (Linear Algebra)
Level Pac-Man (1980), 5 of 10
University of Vermont, Spring 2015



Dispersed: Thursday, February 19, 2015.
Due: By start of lecture, Thursday, February 26, 2015.
Sections covered: 3.1–3.5, some of 3.6.

Some useful reminders:
Instructor: Prof. Peter Dodds
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E-mail: peter.dodds@uvm.edu
Office hours: 2 to 2:45 pm, Mondays; 3 to 3:45 pm Tuesdays; and 1 to 2:30 pm Wednesdays
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124
Textbook: "Introduction to Linear Algebra" (3rd or 4th edition) by Gilbert Strang (published by Wellesley-Cambridge Press).

- All questions are worth 3 points unless marked otherwise.
- Please use a cover sheet and write your name on the back and the front of your assignment.
- You must show all your work clearly.
- You may use Matlab to check your answers for non-Matlab questions (usually Qs. 1–8).
- Please list the names of other students with whom you collaborated.

1. For each of the following reduced row echelon forms of some original matrices, write down the following: m, n, r, the dimension of nullspace, and the dimension of column space, and the number of possible solutions (0, 1, or ∞) depending on $\vec{b}$:

$$\textbf{(a) } \mathbf{R_A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \textbf{(b) } \mathbf{R_A} = \begin{bmatrix} 1 & -2 & 0 & 3 & -1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\textbf{(c) } \mathbf{R_A} = \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Consider a matrix A which is given by the outer product $A = \vec{u}\vec{v}^{\mathrm{T}}$ where

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}.$$

   (a) Find m, n, the rank r for A.

(b) Find column space $C(A)$ and a basis for $C(A)$.

(c) Find nullspace $N(A)$ and a basis for $N(A)$.

(Note: this kind of matrix built from an outer product appears everywhere in real world problems; we'll see more of them later in the semester; you may need to lie down for a while to digest this thrilling detail about your future.)

3. Give all possible forms of $\mathbf{R_A}$, if any exist, for all matrices satisfying the following conditions (use apples, campfires, whatever you like, for any unknowns).

   (a) Rank = 4, dimension of nullspace = 0, $m = 4$.

   (b) Dimension of column space = 5, $n = 5$, $m = 4$.

   (c) Rank = 2, dimension of nullspace = 2, $m = 3$.

   Please assume that the first column is always a pivot column.

4. **(a)** What is the row reduced form $\mathbf{R_A}$ of a 3 by 4 matrix A which has -1 in every entry?

   (b) What are the dimensions of A's column space and nullspace?

   (c) Write down a basis for column space.

5. If $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ are independent vectors, show that the differences $\vec{v}_1 = \vec{w}_2 - \vec{w}_3$, $\vec{v}_2 = \vec{w}_1 - \vec{w}_3$, and $\vec{v}_3 = \vec{w}_1 - \vec{w}_2$ are dependent. Do this by finding a combination of the $\vec{v}$'s that gives $\vec{0}$.

6. Determine whether or not these vectors are independent or dependent: $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

   (Hint: you can test for dependence by placing vectors as rows in a matrix and performing row reduction. Or you can determine if a matrix with these vectors as its columns has a non-trivial nullspace or not.)

7. Suppose $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ are vectors in R^3. Complete the following sentences:

   (a) These four vectors are dependent because ____.

   (b) The two vectors $\vec{v}_1$ and $\vec{v}_2$ will be dependent if ____.

   (c) The vectors $\vec{v}_1$ and $[\ 0\ 0\ 0\ ]^T$ are dependent because ____.

8. True or false (please give a reason if true and a counter example if false):

   (a) The columns of a matrix are a basis for the column space.

   (b) If a matrix contains a column that is all zeros, the columns are dependent.

   (c) If the columns of a matrix are dependent, so are the rows.

9. Matlab question:

   Taking

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

   use Matlab's rref command to find the following reduced row echelon forms:

   (a) $\mathbf{R_A}$,

   (b) $\mathbf{R_{AA^\mathrm{T}}}$,

   (c) $\mathbf{R_{A^\mathrm{T}A}}$,

   and write down m, n, and rank r for all three.

10. Matlab question:

    Consider the n by n family of matrices $A(k;n)$ where $a_{ij} = (i-j)^k$, and k is an integer.

    These matrices are a special kind of weight matrix where the entries increase in magnitude as a function of "distance" from the main diagonal.

    Using Matlab's command "rank", and some experimentation for small k and n, determine how the rank of $A(k;n)$ for general k and n.

    Here's a small function for generating $A(k;n)$. Create an empty file called weightmatrix.m and dump this text in:

```
------------------------------------------------
function A = weightmatrix(k,n)

A = (ones(n,1)*(0:n-1) - (0:n-1)'*ones(1,n)).^k;
------------------------------------------------
```

    See if you can figure out how the above line of code works. The insides contain two outer products. What do they make?

    Show an example weight matrix with $k = 3$ and $n = 5$:

```
>> weightmatrix(3,5)
```

    Find its rank:

```
>> rank(weightmatrix(3,5))
```

    Now play around.

11. Bonus time (1 point):

    What is the lyrebird extremely good at doing?

    See if you can find the David Attenborough BBC video online.