1. (3 + 3 + 3 + 3) This question is all about pure finite and infinite random networks.  
   We’ll define a finite random network as follows. Take $N$ labelled nodes and add  
   links between each pair of nodes with probability $p$.

   (a)  
   i. For a random node $i$, determine the probability distribution for its  
       number of friends $k$, $P_k(p, N)$.
   ii. What kind of distribution is this?
   iii. What does this distribution tend toward in the limit of large $N$, if $p$ is fixed?  
       (No need to do calculations here; just invoke the right Rule of the Universe.)

   (b) Using $P_k(p, N)$, determine the average degree. Does your answer seem right  
       intuitively?

   (c) Show that in the limit of $N \to \infty$ but with mean held constant, we obtain a  
       Poisson degree distribution.  
       Hint: to keep the mean constant, you will need to change $p$.

   (d)  
   i. Compute the clustering coefficients $C_1$ and $C_2$ for standard random  
       networks.
   ii. Explain how your answers make sense.
   iii. What happens in the limit of an infinite random network with finite  
       mean?
2. \((3 + 3)\)

Determine the clustering coefficient for toy model small-world networks \([1]\) as a function of the rewiring probability \(p\). Find \(C_1\), the average local clustering coefficient:

\[
C_1(p) = \left\langle \sum_{j_1, j_2 \in \mathcal{N}_i} a_{j_1 j_2} / k_i(k_i - 1)/2 \right\rangle = \frac{1}{N} \sum_{i=1}^{N} \sum_{j_1, j_2 \in \mathcal{N}_i} a_{j_1 j_2} / k_i(k_i - 1)/2
\]

where \(N\) is the number of nodes, \(a_{ij} = 1\) if nodes \(i\) and \(j\) are connected, and \(\mathcal{N}_i\) indicates the neighborhood of \(i\).

As per the original model, assume a ring network with each node connected to a fixed, even number \(m\) local neighbors \((m/2\) on each side). Take the number of nodes to be \(N \gg m\).

Start by finding \(C_1(0)\) and argue for a \((1 - p)^3\) correction factor to find an approximation of \(C_1(p)\).

Hint 1: you can think of finding \(C_1\) as averaging over the possibilities for a single node.

Hint 2: assume that the degree of individual nodes does not change with rewiring but rather stays fixed at \(m\). In other words, take the average degree of individuals as the degree of a randomly selected individual.

For what value of \(p\) is \(C_1 \approx 1/2\)?

(3 points for set up, 3 for solving.)

References