1. (3 + 3) The 1-d percolation problem:

Consider an infinite 1-d lattice forest with a tree present at any site with probability $p$.

- Find the distribution of forest sizes as a function of $p$. Do this by moving along the 1-d world and figuring out the probability that any forest you enter will extend for a total length $\ell$.

- Find $p_c$, the critical probability for which a giant component exists.
  
  Hint: One way to find critical points is to determine when certain average quantities explode. Compute $\langle l \rangle$ and find $p$ such that this expression goes boom (if it does).

2. Show analytically that the critical probability for site percolation on a triangular lattice is $p_c = 1/2$. 
Hint—Real-space renormalization gets it done.

Direct link: http://www.youtube.com/v/JlkbU5U7QqU?rel=0

3. \((3 + 3)\)

**Coding, it’s what’s for breakfast:**

(a) Percolation in two dimensions \((2-d)\) provides a classic, nutritious example of a phase transition.

Your mission, whether or not you choose to accept it, is to code up and analyse the \(L\) by \(L\) square lattice percolation model for varying \(L\).

Take \(L = 20, 50, 100, 200, 500, \text{ and } 1000\).

(Go higher if you feel \(L = 1000\) is for mere mortals.)

(Go lower if your code explodes.)

Let’s continue with the tree obsession. A site has a tree with probability \(p\), and a sheep grazing on what’s left of a tree with probability \(1 - p\).

Forests are defined as any connected component of trees bordered by sheep, where connections are possible with a site’s four nearest neighbors on a lattice.

Do not bagelize (or doughnutize) the landscape (no periodic boundary conditions—boundaries are boundaries).

(Note: this set up is called site percolation. Bond percolation is the alternate case when all links between neighboring sites exist with probability \(p\).)

Steps:
i. For each $L$, run $N_{\text{tests}}=100$ tests for occupation probability $p$ moving from 0 to 1 in increments of $10^{-2}$. (As for $L$, use a smaller increment if that’s just how you do things.)

ii. Determine the fractional size of the largest connected forest for each of the $N_{\text{tests}}$, and find the average of these, $S_{\text{avg}}$.

iii. On a single figure, for each $L$, plot the average $S_{\text{avg}}$ as a function of $p$.

(b) Comment on how $S_{\text{avg}}(p; N)$ changes as a function of $L$ and estimate the critical probability $p_c$ (the percolation threshold).

Helpful reuse of code (intended for black and white image analysis): You can use Matlab’s bwconncomp to find the sizes of components. Very nice.

4. (3 + 3)

(a) Using your model from the previous question and your estimate of $p_c$, plot the distribution of forest sizes for $p \approx p_c$ for the largest $L$ your code and psychological makeup can withstand. (You can average the distribution over separate simulations.)

Comment on what kind of distribution you find.

(b) Repeat the above for $p = p_c / 2$ and $p = p_c + (1 - p_c) / 2$, i.e., well below and well above $p_c$.

Produce plots for both cases, and again, comment on what you find.