1. Using Gleeson and Calahane’s iterative equations below, derive the contagion condition for a vanishing seed by taking the limit \( \phi_0 \to 0 \) and \( t \to \infty \).

\[
\phi_{t+1} = \phi_0 + (1 - \phi_0) \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \theta_t (1 - \theta_t)^{k-j} B_{kj},
\]

\[
\theta_{t+1} = \phi_0 + (1 - \phi_0) \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_t (1 - \theta_t)^{k-1-j} B_{kj},
\]

where \( \theta_0 = \phi_0 \), and \( B_{kj} \) is the probability that a degree \( k \) node becomes active when \( j \) of its neighbors are active.

Recall that by contagion condition, we mean the requirements of a random network for macroscopic spreading to occur.

To connect the paper’s model and notation to those of our lectures, given a specific response function \( F \) and a threshold model, the \( B_{kj} \) are given by \( B_{kj} = F(j/k) \).

Allow \( B_{k0} \) to be arbitrary (i.e., not necessarily 0 as for simple threshold functions).

Here’s a graphical hint for the three cases you need to consider as \( \theta_0 \to 0 \):
2. Derive equation 4 in Gleeson and Cahalane (2007) [1]:

\[ C_\ell = \sum_{k=\ell+1}^{\infty} \sum_{j=0}^{\ell} \binom{k-1}{\ell} \binom{\ell}{j} (-1)^{\ell+j} \frac{kP_k}{\langle k \rangle} F \left( \frac{j}{k} \right). \]

3. (9 pts)
   
   (a) Derive equation 6 in Gleeson and Cahalane (2007), which is a second order approximation to the cascade condition for vanishing seeds.

   Here’s an example of how this must work:

   (b) Hence reproduce the dashed analytic curve shown in Figure 1 of their paper.

   (c) Explain why there are jumps in the cascade window outline that do not occur at reciprocals of the integers.

4. (6 pts)
   
   (a) By solving for the fixed points of \( \theta_{t+1} = G(\theta_t; 0) \), reproduce Figure 3 in Gleeson and Cahalane (2007):
at high standard deviation and unstable fixed points respectively the value of and highlights the existence of a discontinuous transition in for certain threshold distributions. This is qualitatively different from a smooth low-continuous transition.

negative-threshold agents act as a natural seed, giving a good approximation to the discontinuous low-

Figure 2 shows on Poisson random graphs with threshold distributions, rather than the Bethe lattices of Watts' update rule and standard RFIM dynamics. This model we extend the approach of consider treelike random graphs with arbitrary degree distributions, including power-law degree distributions, and Eq. 6 of \[ G(\theta_t; 0) \] for an average threshold \( \phi_\ast (= R) \) of 0.371 for \( \langle k \rangle = 1, 2, 3, \ldots, 10. \)

Add the cobweb diagram for a \( \phi_0 = 0 \) seed.

Do this by creating a recursive plotting script in matlab, for example. You can use the following Matlab scripts/data as a basis, and most of the work is done. You'll need to improve the plots with some labels, and interpret them properly. The first function calls the other two.

{http://www.uvm.edu/~pdodds/share/matlab/Gfunction.m}
{http://www.uvm.edu/~pdodds/share/matlab/gleeson_fig3_02.mat}
{http://www.uvm.edu/~pdodds/share/matlab/cobweb3.m}

(c) Discuss how the stable points move with \( \langle k \rangle. \)

Note: \( \phi_\ast = 0.371 \) matches plot (b) in Figure 3 of [1].

References