1. Generating functions and giant components: In this question, you will use
   generating functions to obtain a number of results we found in class for standard
   random networks.

   (a) For an infinite standard random network (Erdős-Rényi/ER network) with
       average degree $\langle k \rangle$, compute the generating function $F_P$ for the degree
       distribution $P_k$.

       (Recall the degree distribution is Poisson: $P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$, $k \geq 0$.)

   (b) Show that $F_P'(1) = \langle k \rangle$ (as it should).

   (c) Using the joyous properties of generating functions, show that the second
       moment of the degree distribution is $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.

2. (a) Continuing on from Q1 for infinite standard random networks, find the
   generating function $F_R(x)$ for the $\{R_k\}$, where $R_k$ is the probability that a
   node arrived at by following a random direction on a randomly chosen edge
   has $k$ outgoing edges.

   (b) Now, using $F_R(x)$ determine the average number of outgoing edges from a
       randomly-arrived-at-long-a-random-edge node.

   (c) Given your findings above, what is the condition on $\langle k \rangle$ for a standard
       random network to have a giant component?
3. (a) Find the generating function for the degree distribution $P_k$ of a finite random network with $N$ nodes and an edge probability of $p$.

(b) Show that the generating function for the finite ER network tends to the generating function for the infinite one. Do this by taking the limit $N \to \infty$ and $p \to 0$ such that $p(N - 1) = \langle k \rangle$ remains constant.

4. (a) Prove that if random variables $U$ and $V$ are distributed over the non-negative integers then the generating function for the random variable $W = U + V$ is

$$F_W(x) = F_U(x)F_V(x).$$

Denote the specific distributions by $\Pr(U = i) = U_i$, $\Pr(V = i) = V_i$, and $\Pr(W = i) = W_i$.

(b) Using the your result in part (a), argue that if

$$W = \sum_{j=1}^{U} V^{(j)}$$

where $V^{(j)} \overset{d}{=} V$ then

$$F_W(x) = F_U(F_V(x)).$$

Hint: write down the generating function of probability distribution of $\sum_{j=1}^{k} V^{(j)}$ in terms of $F_V(x)$.

5. (a) Again, given

$$W = \sum_{i=1}^{U} V^{(i)} \text{ with each } V^{(i)} \overset{d}{=} V$$

where we know that

$$F_W(x) = F_U(F_V(x)),$$

determine the mean of $W$ in terms of the means of $U$ and $V$.

(b) For $W = U + V$, similarly find the mean of $W$ in terms of $U$ and $V$ via generating functions. Your answer should make sense.