Mechanisms for Generating Power-Law Size Distributions, Part 1

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Outline

Random Walks
The First Return Problem
Examples

Variable transformation
Basics
Holtsmark’s Distribution
PLIPLO

References
Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time $t$ is $\epsilon_t$:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$
A few random random walks:
Random walks:

Displacement after $t$ steps:

\[ x_t = \sum_{i=1}^{t} \epsilon_i \]

Expected displacement:

\[ \langle x_t \rangle = \langle \sum_{i=1}^{t} \epsilon_i \rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0 \]

- At any time step, we ‘expect’ our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?
Variances sum: (⊞)*

\[
\text{Var}(x_t) = \text{Var} \left( \sum_{i=1}^{t} \epsilon_i \right)
\]

\[
= \sum_{i=1}^{t} \text{Var} (\epsilon_i) = \sum_{i=1}^{t} 1 = t
\]

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

\[
\sigma = t^{1/2}
\]

- A non-trivial scaling law arises out of additive aggregation or accumulation.
Stock Market randomness:

Also known as the bean machine (⊞), the quincunx (simulation) (⊞), and the Galton box.
Great moments in Televised Random Walks:

Plinko! ( salarié) from the Price is Right.
Random walk basics:

Counting random walks:

- Each **specific** random walk of length $t$ appears with a chance $1/2^t$.
- We’ll be more interested in how many random walks end up at the same place.
- Define $N(i,j,t)$ as the number of distinct walks that start at $x = i$ and end at $x = j$ after $t$ time steps.
- Random walk must displace by $(j - i)$ after $t$ steps.
- **Insert question from assignment 2**

$$N(i,j,t) = \binom{t}{(t + j - i)/2}$$
How does $P(x_t)$ behave for large $t$?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \ldots, \pm 2n\}$
- $x_{2n}$ is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have

$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large $n$, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\Pr(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$ 

Insert question from assignment 2 (⊞)

- The whole is different from the parts. #nutritious
- See also: Stable Distributions (⊞)
Universality (brero) is also not left-handed:

- This is **Diffusion (brero)**: the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.
Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step $t$, a random walk has crossed the origin $r$ times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = $\infty$!

See Feller, Intro to Probability Theory, Volume I[3]
Random walks  #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after \( t \) steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We’ll start to see how different scalings relate to each other.
A return to origin can only happen when $t = 2n$.

In example above, returns occur at $t = 8, 10, \text{ and } 14$.

Call $P_{fr}(2n)$ the probability of first return at $t = 2n$.

Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).

Idea: Transform first return problem into an easier return problem.
- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{fr}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n - 1, \text{ and } x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$. 
Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, \( N(i, j, t) \) is the \# of possible walks between \( x = i \) and \( x = j \) taking \( t \) steps.
- Consider all paths starting at \( x = 1 \) and ending at \( x = 1 \) after \( t = 2n - 2 \) steps.
- Idea: If we can compute the number of walks that hit \( x = 0 \) at least once, then we can subtract this from the total number to find the ones that maintain \( x \geq 1 \).
- Call walks that drop below \( x = 1 \) excluded walks.
- We’ll use a method of images to identify these excluded walks.
Examples of excluded walks:

Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of $t$-step paths starting and ending at $x=1$ and hitting $x=0$ at least once = # of $t$-step paths starting at $x=-1$ and ending at $x=1 = N(-1, 1, t)$
- So $N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$
Probability of first return:

Insert question from assignment 2 (توقع غريب):

- Find

\[ N_{fr}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}} \]

- Normalized number of paths gives probability.
- Total number of possible paths = \(2^{2n}\).

\[ P_{fr}(2n) = \frac{1}{2^{2n}} N_{fr}(2n) \]

\[ \approx \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}} \]

\[ = \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}. \]
First Returns

\[ P(t) \propto t^{-3/2}, \gamma = 3/2 \]

- Same scaling holds for continuous space/time walks.
- \( P(t) \) is normalizable.
- **Recurrence**: Random walker always returns to origin
  - But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- **One moral**: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions (\( d \)):  
- Walker in \( d = 2 \) dimensions must also return
- Walker may not return in \( d \geq 3 \) dimensions
Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability.
- Call this probability the Invariant Density of a dynamical system.
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree.
- Equal probability still present: walkers traverse edges with equal frequency.

#groovy
#totallygroovy
Scheidegger Networks \cite{8,2}

- Random directed network on triangular lattice.
- Toy model of real networks.
- ‘Flow’ is southeast or southwest with equal probability.
Scheidegger networks

- Creates basins with random walk boundaries.
- **Observe** that subtracting one random walk from another gives random walk with increments:

\[
\epsilon_t = \begin{cases} 
  +1 & \text{with probability } 1/4 \\
  0 & \text{with probability } 1/2 \\
  -1 & \text{with probability } 1/4
\end{cases}
\]

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length \( \ell \) distribution: \( P(\ell) \propto \ell^{-3/2} \)
- For real river networks, generalize to \( P(\ell) \propto \ell^{-\gamma} \).
Connections between exponents:

- For a basin of length $l$, width $\propto l^{1/2}$
- Basin area $a \propto l \cdot l^{1/2} = l^{3/2}$
- Invert: $l \propto a^{2/3}$
- $dl \propto d(a^{2/3}) = 2/3a^{-1/3}da$
- $\text{Pr(}\text{basin area }= a\text{)}da = \text{Pr(}\text{basin length }= l\text{)}dl$
  $\propto l^{-3/2}dl$
  $\propto (a^{2/3})^{-3/2}a^{-1/3}da$
  $= a^{-4/3}da$
  $= a^{-\tau}da$
Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack’s law \[^4\] :
  \[ \ell \propto a^h. \]
- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (later: allometry).
- Models exist with interesting values of $h$.
- Plan: Redo calc with $\gamma$, $\tau$, and $h$. 

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$
Connections between exponents:

- Given

\[ \ell \propto a^h, \quad P(a) \propto a^{-\tau}, \quad \text{and} \quad P(\ell) \propto \ell^{-\gamma} \]

- \[ d\ell \propto d(a^h) = ha^{h-1}da \]

- Find \( \tau \) in terms of \( \gamma \) and \( h \).

- \[ P(\text{basin area} = a)da = P(\text{basin length} = \ell)d\ell \]

- \[ \propto \ell^{-\gamma}d\ell \]

- \[ \propto (a^h)^{-\gamma}a^{h-1}da \]

- \[ = a^{-(1+h(\gamma-1))}da \]

- \[ \tau = 1 + h(\gamma - 1) \]

- Excellent example of the **Scaling Relations** found between exponents describing power laws for many systems.
Connections between exponents:

With more detailed description of network structure,
\[ \tau = 1 + h(\gamma - 1) \] simplifies to:\footnote{1}

\[ \tau = 2 - h \]

and

\[ \gamma = 1/h \]

- Only one exponent is independent (take \( h \)).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize \textbf{Universality (푀)} class with independent exponents.
Other First Returns or First Passage Times:

Failure:
- A very simple model of failure/death: \([10]\)
- \(x_t = \) entity’s ‘health’ at time \(t\)
- Start with \(x_0 > 0\).
- Entity fails when \(x\) hits 0.

Streams
- Dispersion of suspended sediments in streams.
- Long times for clearing.
More than randomness

- Can generalize to Fractional Random Walks \([6, 7, 5]\)
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: \([5]\)
  “On 1/f noise and other distributions with long tails.”
- In 1-d, standard deviation \(\sigma\) scales as

\[\sigma \sim t^\alpha\]

\(\alpha = 1/2\) — diffusive
\(\alpha > 1/2\) — superdiffusive
\(\alpha < 1/2\) — subdiffusive

- Extensive memory of path now matters...
Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable $X$ with known distribution $P_x$
- Second random variable $Y$ with $y = f(x)$.

$$P_y(y)dy = P_x(x)dx$$

$$= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|}$$

- Often easier to do by hand...
General Example

- Assume relationship between $x$ and $y$ is 1-1.
- Power-law relationship between variables:
  \[ y = cx^{-\alpha}, \, \alpha > 0 \]
- Look at $y$ large and $x$ small

\[
dy = d \left( cx^{-\alpha} \right)
\]

\[
= c(-\alpha)x^{-\alpha-1}dx
\]

**invert:** \[ dx = \frac{-1}{c\alpha} x^{\alpha+1} dy \]

\[
dx = \frac{-1}{c\alpha} \left( \frac{y}{c} \right)^{-(\alpha+1)/\alpha} dy
\]

\[
dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy
\]
Now make transformation:

\[ P_y(y) \, dy = P_x(x) \, dx \]

\[ P_y(y) \, dy = P_x \left( \left( \frac{y}{c} \right)^{-1/\alpha} \right) \frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} \, dy \]

- If \( P_x(x) \to \) non-zero constant as \( x \to 0 \) then
  \[ P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \to \infty. \]

- If \( P_x(x) \to x^\beta \) as \( x \to 0 \) then
  \[ P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty. \]
Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.
Gravity

- Select a random point in the universe \( \vec{x} \)
- Measure the force of gravity \( F(\vec{x}) \)
- Observe that \( P_F(F) \sim F^{-5/2} \).
Matter is concentrated in stars: [9]

- $F$ is distributed unevenly
- Probability of being a distance $r$ from a single star at $\vec{x} = \vec{0}$:
  \[ P_r(r)dr \propto r^2 dr \]
- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:
  \[ F \propto r^{-2} \]
- Invert:
  \[ r \propto F^{-1/2} \]
- Also invert:
  \[ dF \propto d(r^{-2}) \propto r^{-3} dr \rightarrow dr \propto r^3 dF \propto F^{-3/2} dF . \]
Transformation:

Using $r \propto F^{-1/2}$, $\text{d}r \propto F^{-3/2} \text{d}F$, and $P_r(r) \propto r^2$

- $P_F(F) \text{d}F = P_r(r) \text{d}r$

- $\propto P_r(F^{-1/2}) F^{-3/2} \text{d}F$

- $\propto \left(F^{-1/2}\right)^2 F^{-3/2} \text{d}F$

- $= F^{-1-3/2} \text{d}F$

- $= F^{-5/2} \text{d}F$.
Gravity:

\[ P_F(F) = F^{-5/2} \, dF \]

- Mean is finite.
- Variance = \( \infty \).
- A wild distribution.
- **Upshot:** Random sampling of space usually safe but can end badly...
Extreme Caution!

- **PLIPLO** = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (...)
- Don’t do this!!! (slap, slap)
- We need mechanisms!
References I


References II


References III


