1. (3 + 3)

Coding, it’s what’s for breakfast:

(a) Percolation in two dimensions (2-d) provides a classic, nutritious example of a phase transition.

Your mission, whether or not you choose to accept it, is to code up and analyse the $L \times L$ square lattice percolation model for varying $L$.

Take $L = 20, 50, 100, 200, 500, \text{ and } 1000$.

(Go higher if you feel $L = 1000$ is for mere mortals.)

(Go lower if your code explodes.)

Let’s continue with the tree obsession. A site has a tree with probability $p$, and a sheep grazing on what’s left of a tree with probability $1 - p$.

Forests are defined as any connected component of trees bordered by sheep, where connections are possible with a site’s four nearest neighbors on a lattice.

Do not bagelize (or doughnutize) the landscape (no periodic boundary conditions—boundaries are boundaries).

(Note: this set up is called site percolation. Bond percolation is the alternate case when all links between neighboring sites exist with probability $p$.)

Steps:
i. For each $L$, run $N_{\text{tests}}=100$ tests for occupation probability $p$ moving from 0 to 1 in increments of $10^{-2}$. (As for $L$, use a smaller increment if that’s just how you do things.)

ii. Determine the fractional size of the largest connected forest for each of the $N_{\text{tests}}$, and find the average of these, $S_{\text{avg}}$.

iii. On a single figure, for each $L$, plot the average $S_{\text{avg}}$ as a function of $p$.

(b) Comment on how $S_{\text{avg}}(p; N)$ changes as a function of $L$ and estimate the critical probability $p_c$ (the percolation threshold).

Helpful reuse of code (intended for black and white image analysis): You can use Matlab’s bwconncomp to find the sizes of components. Very nice.

2. $(3 + 3)$

(a) Using your model from the previous question and your estimate of $p_c$, plot the distribution of forest sizes for $p \approx p_c$ for the largest $L$ your code and psychological makeup can withstand. (You can average the distribution over separate simulations.)

Comment on what kind of distribution you find.

(b) Repeat the above for $p = p_c/2$ and $p = p_c + (1 - p_c)/2$, i.e., well below and well above $p_c$.

Produce plots for both cases, and again, comment on what you find.

3. Show analytically that the critical probability for site percolation on a triangular lattice is $p_c = 1/2$.

**Hint—Real-space renormalization gets it done:**

http://www.youtube.com/v/JlkB5u7QqU?rel=0

4. In lectures on lognormals and other heavy-tailed distributions, we came across a super fun and interesting integral when considering organization size distributions arising from growth processes with variable lifespans.

Show that

$$P(x) = \int_{t=0}^{\infty} \frac{\lambda e^{-\lambda t}}{x^{\sqrt{2\pi t}}} \exp \left( -\frac{(\ln \frac{x}{m})^2}{2t} \right) dt$$

leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}},$$

and therefore, surprisingly, two different scaling regimes. Enjoyable suffering may be involved. Really enjoyable suffering.

Hints and steps:
• Make the substitution \( t = u^2 \) to find an integral of the form

\[
I_1(a, b) = \int_0^\infty \exp \left( -au^2 - b/u^2 \right) \, du
\]

where in our case \( a = \lambda \) and \( b = (\ln \frac{z}{m})^2/2 \).

• Substitute \( au^2 = t^2 \) into the above to find

\[
I_1(a, b) = \frac{1}{\sqrt{a}} \int_0^\infty \exp \left( -t^2 - ab/t^2 \right) \, dt
\]

• Now work on this integral:

\[
I_2(r) = \int_0^\infty \exp \left( -t^2 - r/t^2 \right) \, dt
\]

where \( r = ab \).

• Differentiate \( I_2 \) with respect to \( r \) to create a simple differential equation for \( I_2 \). You will need to use the substitution \( u = \sqrt{r}/t \) and your differential equation should be of the form

\[
\frac{dI_2(r)}{dr} = -(something) I_2(r).
\]

• Solve the differential equation you find. To find the constant of integration, you can evaluate \( I_2(0) \) separately:

\[
I_2(0) = \int_0^\infty \exp(-t^2) \, dt,
\]

where our friend \( \Gamma(1/2) \) comes into play.

5. (3 + 3 + 3 + 3) This question is all about pure finite and infinite random networks.

We’ll define a finite random network as follows. Take \( N \) labelled nodes and add links between each pair of nodes with probability \( p \).

(a) i. For a random node \( i \), determine the probability distribution for its number of friends \( k \), \( P_k(p, N) \).

ii. What kind of distribution is this?

iii. What does this distribution tend toward in the limit of large \( N \), if \( p \) is fixed?

(No need to do calculations here; just invoke the right Rule of the Universe.)

(b) Using \( P_k(p, N) \), determine the average degree. Does your answer seem right intuitively?
(c) Show that in the limit of $N \to \infty$ but with mean held constant, we obtain a Poisson degree distribution.

Hint: to keep the mean constant, you will need to change $p$.

(d) i. Compute the clustering coefficients $C_1$ and $C_2$ for standard random networks.

ii. Explain how your answers make sense.

iii. What happens in the limit of an infinite random network with finite mean?