Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2013
Assignment 4 • code name: One Does Not Simply Walk into Mordor

Dispersed: Thursday, September 26, 2013.
Due: By start of lecture, 1:00 pm, Thursday, October 3, 2013.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 1:00 pm to 4:00 pm, Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2013-08UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. (3 + 3) The 1-d percolation problem:

   Consider an infinite 1-d lattice forest with a tree present at any site with probability p.

   - Find the distribution of forest sizes as a function of p. Do this by moving along the 1-d world and figuring out the probability that any forest you enter will extend for a total length ℓ.
   - Find p_c, the critical probability for which a giant component exists.
     Hint: One way to find critical points is to determine when certain average quantities explode. Compute $\langle l \rangle$ and find p such that this expression goes boom (if it does).

2. (3 + 3 + 3)

   A courageous coding festival:

   Code up the discrete HOT model in 1-d. Let's see if we find any of these super-duper power laws everyone keeps talking about. We'll follow the same approach as the 2-d forest discussed in lectures.

   Main goal: extract yield curves as a function of the design D parameter as described below.

   Suggested simulations elements:

- $N = 10^4$ as a start. Then see if $N = 10^5$ or $N = 10^6$ is possible.
- Start with no trees.
- Probability of a spark at the ith site: $P(i) \propto e^{-i/\ell}$ where i is tree position ($i = 1$ to N). (You will need to normalize this properly.) The quantity ℓ is the characteristic scale for this distribution; try $\ell = 2 \times 10^5$.
- Consider a design problem of $D = 1$, 2, $N^{1/2}$, and N. (If $N^{1/2}$ and N are too much, you can drop them. Perhaps sneak out to $D = 3$.) Recall that the design problem is to test D randomly chosen placements of the next tree against the spark distribution.
- For each test tree, measure the average yield (number of trees left) with $n = 100$ randomly selected sparks. Select the tree location with the highest average yield and plant a tree there.
- Add trees until the linear forest is full, measuring average yield as a function of trees added.
- Only trees and adjacent trees burn. In effect, you will be burning un-treed intervals of the line (much less complicated than 2-d).

(a) Plot the yield curves for each value of D.

(b) Identify peak yield for each value of D.

(c) Plot distributions of connected tree interval sizes at peak yield (you will have to rebuild forests and stop at the peak yield value of D to find these distributions.

Hint: keeping a list of un-treed locations will make choosing the next location easier. Hopefully.

3. The discrete version of HOT theory:

From lectures, we had the follwoing.

Cost: Expected size of 'fire' in a d-dimensional lattice:

$$C_{\text{fire}} \propto \sum_{i=1}^{N_{\text{sites}}} (p_i a_i) a_i = \sum_{i=1}^{N_{\text{sites}}} p_i a_i^2,$$

where a_i = area of ith site's region, and p_i = avg. prob. of fire at site in ith site's region.

From lectures, the constraint for building and maintaining $(d-1)$-dimensional firewalls in d-dimensions is

$$C_{\text{firewalls}} \propto \sum_{i=1}^{N_{\text{sites}}} a_i^{(d-1)/d} a_i^{-1},$$

where we are assuming isometry.

Using Lagrange Multipliers, safety goggles, rubber gloves, a pair of tongs, and a maniacal laugh, determine that:

$$p_i \propto a_i^{-\gamma} = a_i^{-(2+1/d)}.$$

4. (3 + 3 + 3) *Highly Optimized Tolerance:*

   This question is based on Carlson and Doyle's 1999 paper "Highly optimized tolerance: A mechanism for power laws in design systems" [1]. In class, we made our way through a discrete version of a toy HOT model of forest fires. This paper revolves around the equivalent continuous model's derivation.

   Our interest is in Table I on p. 1415:

$p(x)$	$p_{\mathrm{cum}}(x)$	$P_{\mathrm{cum}}(A)$
$x^{-(q+1)}$	x^{-q}	$A^{-\gamma(1-1/q)}$
e^{-x}	e^{-x}	$A^{-\gamma}$
e^{-x^2}	$x^{-1}e^{-x^2}$	$A^{-\gamma}[\log(A)]^{-1/2}$

   and Equation 8 on the same page:

$$P_{\geq}(A) = \int_{p^{-1}(A^{-\gamma})}^{\infty} p(\mathbf{x})\mathrm{d}\mathbf{x} = p_{\geq}\left(p^{-1}\left(A^{-\gamma}\right)\right),$$

   where $\gamma = \alpha + 1/\beta$ and we'll write $P_{\geq}$ for P_{cum}.

   Please note that $P_{\geq}(A)$ for $x^{-(q+1)}$ is not correct. Find the right one!

   Here, $A(\mathbf{x})$ is the area connected to the point $\mathbf{x}$ (think connected patch of trees for forest fires). The cost of a 'failure' (e.g., lightning) beginning at $\mathbf{x}$ scales as $A(\mathbf{x})^{\alpha}$ which in turn occurs with probability $p(\mathbf{x})$. The function p^{-1} is the inverse function of p.

   Resources associated with point $\mathbf{x}$ are denoted as $R(\mathbf{x})$ and area is assumed to scale with resource as $A(\mathbf{x}) \sim R^{-\beta}(\mathbf{x})$.

   Finally, $p_{\geq}$ is the complementary cumulative distribution function for p.

   As per the table, determine $p_{\geq}(x)$ and $P_{\geq}(A)$ for the following (3 pts each):

   (a) $p(x) = cx^{-(q+1)}$,

   (b) $p(x) = ce^{-x}$, and

   (c) $p(x) = ce^{-x^2}$.

   Note that these forms are for the tails of p only, and you should incorporate a constant of proportionality c, which is not shown in the paper.

References

[1] J. M. Carlson and J. Doyle. Highly optimized tolerance: A mechanism for power laws in designed systems. Phys. Rev. E, 60(2):1412–1427, 1999.