The law of first digits

Benford's Law: \( P(\text{first digit} = d) \propto \log_b \left(1 + \frac{1}{d}\right) \)

for certain sets of ‘naturally’ occurring numbers in base \( b \)

- Around 30.1% of first digits are ‘1’, compared to only 4.6% for ‘9’.
- First observed by Simon Newcomb in 1881 “Note on the Frequency of Use of the Different Digits in Natural Numbers”
- Independently discovered in 1938 by Frank Benford.
- Newcomb almost always noted but Benford gets the stamp.

References

Benford's Law

Observed for

- Fundamental constants (electron mass, charge, etc.)
- Utility bills
- Numbers on tax returns (ha!)
- Death rates
- Street addresses
- Numbers in newspapers
- Cited as evidence of fraud in the 2009 Iranian elections.

Real data:


Physical constants of the universe:

Taken from here.
Benford's Law

Population of countries:

![Bar chart showing population distribution of countries.](image)

Taken from [here](#).

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Essential story

- \( P(\text{first digit} = d) \propto \log_b \left(1 + \frac{1}{d}\right) \)
- \( \propto \log_b \left(\frac{d + 1}{d}\right) \)
- \( \propto \log_b (d + 1) - \log_b (d) \)

- Observe this distribution if numbers are distributed uniformly in log-space:
  \[ P(\ln x) d(\ln x) \propto 1 \cdot d(\ln x) = x^{-1} \, dx \]

- Power law distributions at work again...
- Extreme case of \( \gamma \simeq 1 \).

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