1. (3 + 3)

Using Gleeson and Calahane’s iterative equations below, derive the contagion condition for a vanishing seed by taking the limit $\phi_0 \to 0$ and $t \to \infty$. In lectures, we derived the discrete evolution equations for the fraction of infected nodes $\phi_t$ and the fraction of infected edges $\theta_t$ as follows:

$$
\phi_{t+1} = \phi_0 + (1 - \phi_0) \sum_{k=0}^{\infty} P_k \sum_{j=0}^{k} \binom{k}{j} \theta_j (1 - \theta_t)^{k-j} B_{kj},
$$

$$
\theta_{t+1} = G(\theta_t; \phi_0) = \phi_0 + (1 - \phi_0) \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_j (1 - \theta_t)^{k-1-j} B_{kj},
$$

where $\theta_0 = \phi_0$, and $B_{kj}$ is the probability that a degree $k$ node becomes active when $j$ of its neighbors are active.

Recall that by contagion condition, we mean the requirements of a random network for macroscopic spreading to occur.

To connect the paper’s model and notation to those of our lectures, given a specific response function $F$ and a threshold model, the $B_{kj}$ are given by $B_{kj} = F(j/k)$.

Allow $B_{k0}$ to be arbitrary (i.e., not necessarily 0 as for simple threshold functions).

We really only need to understand how $\theta_t$ behaves. Write the corresponding equation as $\theta_{t+1} = G(\theta_t; \phi_0)$ and determine when
(a) $G(0; \phi_0) > 0$ (spreading is for free).

(b) $G(0; \phi_0) = 0$ and $G'(0; \phi_0) > 1$ meaning $\phi = 0$ is a unstable fixed point.

Here’s a graphical hint for the three cases you need to consider as $\theta_0 \to 0$:

Success: 

Failed:

2. $(3 + 3 + 3)$ More on the power law stuff:

Take $x$ to be the wealth held by an individual in a population of $n$ people, and the number of individuals with wealth between $x$ and $x + dx$ to be approximately $N(x)dx$.

Given a power-law size frequency distribution $N(x) = cx^{-\gamma}$ where $x_{\text{min}} \ll x \ll \infty$, determine the value of $\gamma$ for which the so-called 80/20 rule holds.

In other words, find $\gamma$ for which the bottom 4/5 of the population holds 1/5 of the overall wealth, and the top 1/5 holds the remaining 4/5.

Assume the mean is finite, i.e., $\gamma > 2$.

(a) First determine the total wealth $W$ in the system given $\int_{x_{\text{min}}}^{\infty} dx N(x) = n$.

(b) Find $\gamma$ for the 80/20 requirement.

(c) For the $\gamma$ you find, determine how much wealth $100q\%$ of the population possesses as a function of $q$ and plot the result.

3. $(3 + 3)$

(a) Let’s generalize the preceding question so that $100q\%$ of the population holds $100(1 - r)\%$ of the wealth.

Show $\gamma$ depends on $p$ and $q$ as

$$\gamma = 1 + \frac{\ln \frac{1}{1-q}}{\ln \frac{1}{1-q} - \ln \frac{1}{r}}.$$

(Check this agrees with your result for the previous question by setting $q = 4/5$ and $r = 1/5$.

(b) Is every pairing of $q$ and $r$ possible?