Mechanisms for Generating Power-Law Size Distributions I
Principles of Complex Systems
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Outline

Random Walks
The First Return Problem
Examples

Variable transformation
Basics
Holtsmark's Distribution
PLIPLO

References

Mechanisms

A powerful story in the rise of complexity:
▶ structure arises out of randomness.
▶ Exhibit A: Random walks... (III)

Random walks

The essential random walk:
▶ One spatial dimension.
▶ Time and space are discrete
▶ Random walker (e.g., a drunk) starts at origin \( x = 0 \).
▶ Step at time \( t \) is \( \epsilon_t \):

\[
\epsilon_t = \begin{cases} 
+1 & \text{with probability } 1/2 \\
-1 & \text{with probability } 1/2 
\end{cases}
\]

Displacement after \( t \) steps:

\[
x_t = \sum_{i=1}^{t} \epsilon_i
\]

Expected displacement:

\[
\langle x_t \rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0
\]

Variances sum: (III) *

\[
\text{Var}(x_t) = \text{Var} \left( \sum_{i=1}^{t} \epsilon_i \right) = \sum_{i=1}^{t} \text{Var}(\epsilon_i) = \sum_{i=1}^{t} 1 = t
\]

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.
Random walks

So typical displacement from the origin scales as

\[ \sigma = t^{1/2} \]

\[ \Rightarrow \text{A non-trivial power-law arises out of additive aggregation or accumulation.} \]

Random walks

Random walks are weirder than you might think...

For example:

- \( \xi_{r,t} \) is the probability that by time step \( t \), a random walk has crossed the origin \( r \) times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.

See Feller, [2] Intro to Probability Theory, Volume I

Random walks

In fact:

\[ \xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots \]

Even crazier:

The expected time between tied scores = \( \infty \)!

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after \( t \) steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?
First returns

Reasons for caring:
1. We will find a power-law size distribution with an interesting exponent
2. Some physical structures may result from random walks
3. We’ll start to see how different scalings relate to each other

For random walks in 1-d:
- Return can only happen when \( t = 2n \).
- Call \( P_{\text{first return}}(2n) = P_u(2n) \) probability of first return at \( t = 2n \).
- Assume drunkard first lurches to \( x = 1 \).
- The problem
  \[
  P_u(2n) = 2 \Pr(x_t \geq 1, t = 1, \ldots, 2n - 1, \text{ and } x_{2n} = 0)
  \]
- A useful restatement: \( P_u(2n) = 2 \cdot \frac{1}{2} \Pr(x_t \geq 1, t = 1, \ldots, 2n - 1, \text{ and } x_1 = x_{2n-1} = 1) \)
- Want walks that can return many times to \( x = 1 \).
- (The \( \frac{1}{2} \) accounts for stepping to 2 instead of 0 at \( t = 2n \).)

Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define \( N(i, j, t) \) as the # of possible walks between \( x = i \) and \( x = j \) taking \( t \) steps.
- Consider all paths starting at \( x = 1 \) and ending at \( x = 1 \) after \( t = 2n - 2 \) steps.
- Subtract how many hit \( x = 0 \).
First Returns

Key observation:

# of t-step paths starting and ending at \( x = 1 \)
and hitting \( x = 0 \) at least once

= # of t-step paths starting at \( x = -1 \) and ending at \( x = 1 \)

= \( N(-1, 1, t) \)

So \( N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2) \)

See this 1-1 correspondence visually...

Next problem: what is \( N(i, j, t) \)?

- # positive steps + # negative steps = \( t \).
- Random walk must displace by \( j - i \) after \( t \) steps.
- # positive steps - # negative steps = \( j - i \).
- # positive steps = \( (t + j - i) / 2 \).

\[ N(i, j, t) = \left( \frac{t}{\# \text{ positive steps}} \right) = \left( \frac{t + j - i}{2} \right) \]

We now have

\[ N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2) \]

where

\[ N(i, j, t) = \left( \frac{t}{(t + j - i)/2} \right) \]
First Returns

Insert question from assignment 4 (III)

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}$

- Normalized Number of Paths gives Probability
- Total number of possible paths $= 2^{2n}$

$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

$$\simeq \frac{1}{2^{2n}} 2^{2n-3/2} \sqrt{2\pi n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2}$$

Random walks

On finite spaces:
- In any finite volume, a random walker will visit every site with equal probability
- Random walking $\equiv$ Diffusion
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

First Returns

- Same scaling holds for continuous space/time walks.
- $$P(t) \propto t^{-3/2}, \gamma = 3/2$$
- $P(t)$ is normalizable
- **Recurrence**: Random walker always returns to origin
- **Moral**: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Random walks on

On networks:
- On networks, a random walker visits each node with frequency $\propto$ node degree
- Equal probability still present: walkers traverse edges with equal frequency.

First Returns

Higher dimensions:
- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...

Scheidegger Networks [4, 1]

- Triangular lattice
- 'Flow' is southeast or southwest with equal probability.
Both basin area and length obey power law distributions.

- Observed for real river networks
- Typically: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$
- Smaller basins more allometric ($h > 1/2$)
- Larger basins more isometric ($h = 1/2$)

Redo calc with $\gamma$, $\tau$, and $h$.

Given $\ell \propto a^h$, $P(a) \propto a^{-\tau}$, and $P(\ell) \propto \ell^{-\gamma}$,

$$d\ell \propto d(a^\gamma) = h a^{h-1} da$$

$$Pr(\text{basin area} = a) da = Pr(\text{basin length} = \ell) d\ell$$

$$\propto \ell^{-\gamma} d\ell$$

$$\propto a^{h-1} da$$

$$= a^{-h+1} da$$

$$= a^{(1+\gamma-1)} da$$

$$\tau = 1 + h(\gamma - 1)$$

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

Only one exponent is independent

Simplify system description

Expect scaling relations where power laws are found

Characterize universality class with independent exponents
Other First Returns

Failure
- A very simple model of failure/death:
- \( x_t \) = entity’s ‘health’ at time \( t \)
- \( x_0 \) could be \( > 0 \).
- Entity fails when \( x \) hits 0.

Streams
- Dispersion of suspended sediments in streams.
- Long times for clearing.

More than randomness

- Can generalize to Fractional Random Walks
- Levy flights, Fractional Brownian Motion
- In 1-d,
  \[ \sigma \sim t^\alpha \]
  \( \alpha > 1/2 \) — superdiffusive
  \( \alpha < 1/2 \) — subdiffusive
- Extensive memory of path now matters...

Variable Transformation

Understand power laws as arising from
1. elementary distributions (e.g., exponentials)
2. variables connected by power relationships

Variable Transformation

Random variable \( X \) with known distribution \( P_X \)
Second random variable \( Y \) with \( y = f(x) \).

\[ P_Y(y)dy = P_X(x)dx \]

\[ = \sum_{y'(f^{-1}(y))=y} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|} \]

Often easier to do by hand...

General Example

Assume relationship between \( x \) and \( y \) is 1-1.
- Power-law relationship between variables:
  \( y = cx^{-\alpha}, \alpha > 0 \)
- Look at \( y \) large and \( x \) small

\[ dy = d\left(cx^{-\alpha}\right) = c(-\alpha)x^{-\alpha-1}dx \]

\[ \text{invert: } dx = \frac{1}{c\alpha}y^{\alpha+1}dy \]

\[ dx = \frac{1}{c\alpha}y^{-(\alpha+1)/\alpha}dy \]

\[ dx = \frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}dy \]
Example

**Exponential distribution**

Given \( P_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \) and \( y = c x^{-\alpha} \), then

\[
P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})
\]

- Exponentials arise from randomness...
- More later when we cover robustness.

Gravity

- Select a random point in the universe \( \vec{x} \)
- (possible all of space-time)
- Measure the force of gravity \( F(\vec{x}) \)
- Observe that \( P_F(F) \sim F^{-5/2} \).

**Ingredients**

- Matter is concentrated in stars:
  - \( F \) is distributed unevenly
  - Probability of being a distance \( r \) from a single star at \( \vec{x} = 0 \):
    \[
P_r(r)dr \propto r^2 dr
    \]
  - Assume stars are distributed randomly in space (oops?)
  - Assume only one star has significant effect at \( \vec{x} \).
  - Law of gravity:
    \[
    F \propto r^{-2}
    \]
  - **invert:**
    \[
    r \propto F^{-1/2}
    \]

Transformation

\[
\begin{align*}
\frac{dF}{dr} & \propto r^{-2} \\
\propto r^{-3} dr \\
\text{invert:} & \\
dr & \propto r^3 dF \\
& \propto F^{-3/2} dF
\end{align*}
\]

Using \( r \propto F^{-1/2} \) and \( P_r(r) \propto r^2 \)

\[
\begin{align*}
P_F(F) & = P_r(r)dr \\
& \propto P_r(F^{-1/2}) F^{-3/2} dF \\
& \propto (F^{-1/2})^2 F^{-3/2} dF \\
& = F^{-1-3/2} dF \\
& = F^{-5/2} dF
\end{align*}
\]

\( \gamma = 5/2 \)

- Mean is finite
- Variance = \( \infty \)
- **A wild distribution**
- Random sampling of space usually safe but can end badly...
Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (⊞)... Don't do this!!! (slap, slap)
- We need mechanisms!

References I


References II
