# LINEAR ALGEBRA

## Vector Spaces (still all about $\mathbb{A}\vec{x}=\vec{b}$)

The column picture again:

$$\mathbb{A}=\begin{bmatrix}3&1&-1\\0&1&0\\2&0&4\end{bmatrix}\qquad \vec{b}=\begin{bmatrix}2\\1\\3\end{bmatrix}$$

we Solve $\mathbb{A}\vec{x}=\vec{b}$ for $\vec{x}$ to find out how we combine $\vec{a}_1, \vec{a}_2, \vec{a}_3$ column vectors of $\mathbb{A}$ $\left(\begin{bmatrix}3\\0\\2\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\4\end{bmatrix}\right)$ to create/generate/reach $\vec{b}$.

We need to understand the places (spaces) where $\vec{a}_i$'s, $\vec{x}$'s, & $\vec{b}$'s live

(where they can be, where they can't be)

Skip this

~~Let's start~~ with the real numbers $\mathbb{R}$ which is the essential vector space we'll focus on.

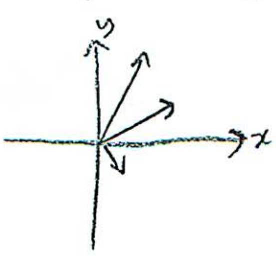
ey aspects of $\mathbb{R}$

) if $x, y \in \mathbb{R}$ then $x+y \in \mathbb{R}$

[a plain observation
$3+5=8$
$3\times 7=21$

) if ~~we~~ multiply $x$ by $c\in\mathbb{R}$, $cx$ is still in $\mathbb{R}$

Let's think about all vectors of ~~length~~ in 2-d 2 : $\mathbb{R}^2=\left\{\begin{bmatrix}x\\y\end{bmatrix};\ x,y\in\mathbb{R}\right\}$



the idealized plane

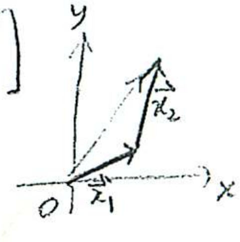
(somewhat obvious)

Two key aspects of a vector space $\mathbb{V}$:

① If we add any two vectors in $\mathbb{V}$, we get another vector that still lives in $\mathbb{V}$

$$\begin{bmatrix}3\\4\end{bmatrix}+\begin{bmatrix}773\\2\end{bmatrix}=\begin{bmatrix}776\\6\end{bmatrix}\quad \leftarrow \text{e.g. } \mathbb{R}^2$$

$$\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}$$



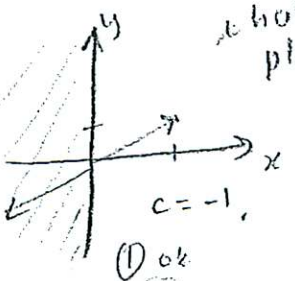
② If we multiply a vector in $\mathbb{V}$ by any real number $c\in\mathbb{R}$, then the new vector still lives in $\mathbb{V}$.

e.g. $7\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}7\cdot 3\\7\cdot 4\end{bmatrix}=\begin{bmatrix}21\\28\end{bmatrix}$ ✓

$c\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}cx_1\\cy_1\end{bmatrix}$ ✓

* make into question

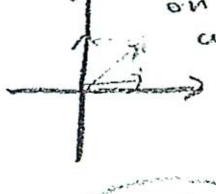
[① & ② ⇒ Vector spaces are closed under addition and scalar multiplication



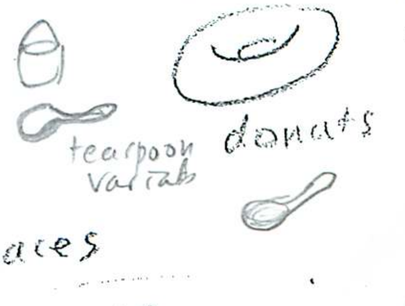
a half plane

$c=-1$.

① ok
② ×



all pts on x, y axes.



↑ examples of non-vector spaces

| 3  1 2  space

# LINEAR ALGEBRA

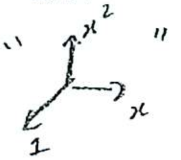
'Ano interesting vector space

|| polynomials of degree $n$.
or less

a) e.g. $n=2$

$f_1(x) = 2 + 3x + 6x^2$

$f_2(x) = -1 + 2x - 3x^2$

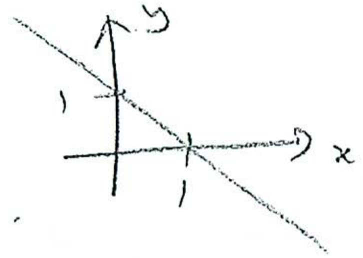


$f_1(x) + f_2(x) = 1 + 5x + 3x^2$ ✓

b) What about all $m \times n$ matrices ?

Yes!

c) what about all points on the line $x + y = 1$



d) what about the integers?

**General Requirements of a vector Space**

V is a set of "vectors"
some kind of element

such that the following hold

(VS P1) if $\vec{x}, \vec{y} \in V$ then $\vec{x} + \vec{y} \in V$

(VS P2) if $\vec{x} \in V$, $c\vec{x} \in V$ for all $c \in \mathbb{R}$

(VS P3) $\vec{0} \in V$ $(\vec{0} + \vec{x} = \vec{x})$

↓ increasingly boring properties

eg $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$

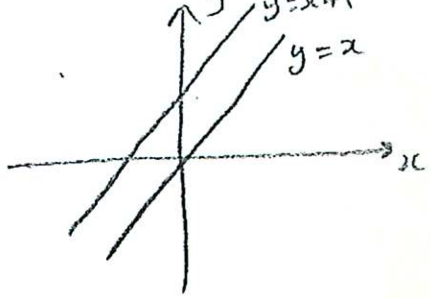
* We are most interested in these first three conditions

\+ We focus solely on $\mathbb{R}^n$
$n = 0, 1, 2, 3$

Super crucial observation:
Vector spaces have smaller vector space living inside them.

We call these Subspaces

e.g.



$\mathbb{R}^2$ is main space

- the vectors in the line $y = x$ form a subspace of $\mathbb{R}^2$
- those in the line $y = x + 1$ do not.

Reason: we want subspaces to behave like vector spaces

(PSS1) if $\vec{x} + \vec{y} \in S$

(PSS2) if $\vec{x} \in S$, $c\vec{x} \in S$

(PSS3) $\vec{0} \in S$

So while $y = x + 1$ is a nice $\infty$ line that looks like $\mathbb{R}$, it does not satisfy any of the properties PSS1 – PSS2

**Show** n.b. $\{\vec{0}\}$ & $\mathbb{R}^2$ are both subspaces of $\mathbb{R}^2$

Do the integers form a subspace of $\mathbb{R}$

What about the rationals? [illegible] ?

conceptual as

# LINEAR ALGEBRA

For $\mathbb{R}^3$, subspaces are $\mathbb{R}^3$ itself, $\{\vec{0}\}$, any plane passing through the origin, & any line passing through the origin.

We've talked about $R, R^2, R^3$

In general we have $\mathbb{R}^m$ $n = 1, 2, 3, 4, 5, \ldots$ and a vector $\vec{b}$ lives in $\mathbb{R}^m$ if it has $n$ entries

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

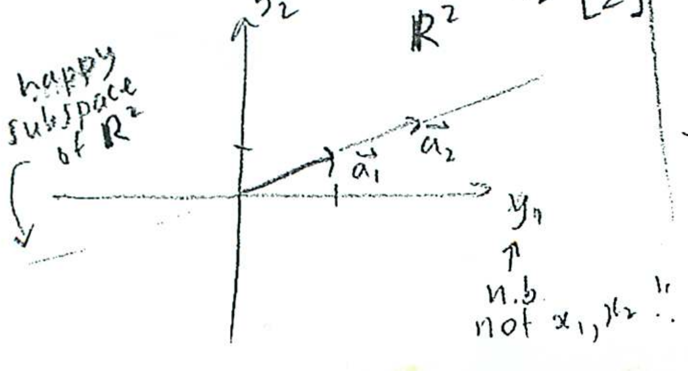
Our beloved problem $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$

column picture:

$$\underbrace{\vec{a}_1 x_1 + \vec{a}_2 x_2 + \cdots + \vec{a}_n x_n}_{} = \vec{b}$$

columns of $A$ also live in $\mathbb{R}^m$ (n.b. not $\mathbb{R}^n$ !!!)

eg. $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$, $\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ =



# $C(A)$

## L ch3 3

$$\left\{ \vec{y} \,\middle|\, \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 2 \end{bmatrix} x_2 \right\}, \ x_1, x_2 \in R$$

$C(A)$ = span of those vectors

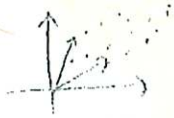
means: take all linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ ← huge!!

The big deal:

$A\vec{x} = \vec{b}$ has a solution (possibly 1 or ∞ly many) if & only if $\vec{b}$ lives in $A$'s column space.

For $A$ above, $\vec{b} = \begin{bmatrix} 38 \\ 19 \end{bmatrix}$ works

$\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ fails



eg. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

now $C(A) = R^2$ any $\vec{b}$ works!

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

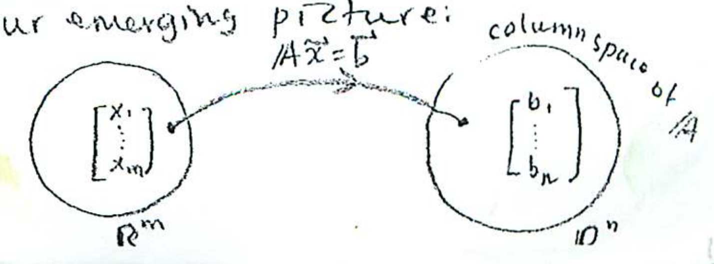
draw pics

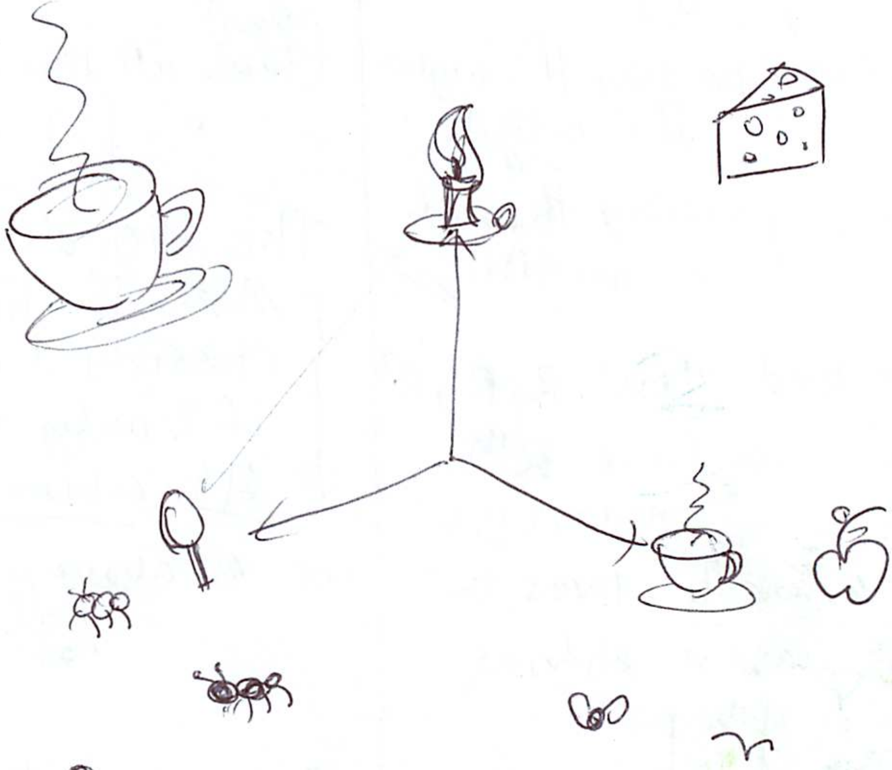
$C(A)$ is a plane (2-d) living in 3-d

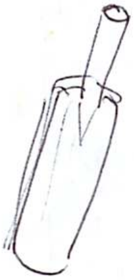
What about $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ ?

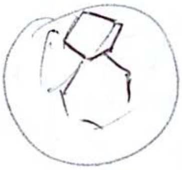
example of finding column space soon

Our emerging picture:









19 pages

# Linear Algebra

Sec 2.7
Transposes (& permutations)
Sec's 3.1–3.6
Sec's 4.1–4.4 (maybe not so much 4.4)
Chapter 5
Chapter 6

[SVD post thanksgiving]

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## Section 3.2

### The Nullspace of $A$

some kind of vector space...

Consider $A\vec{x} = \vec{0}$ ← zero vector (special $\vec{b}$)
(We call this a homogeneous equation.)

→ How can we combine the columns of $A$ to obtain $\vec{0}$?

$\vec{x} = \vec{0}$ always works!
but consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} +1 \\ +1 \\ -1 \end{bmatrix}$$

eyeballing

in fact, any multiple of this works

So $A\vec{x} = \vec{0}$ for all $\vec{x}_h = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ← subspace of $\mathbb{R}^3$

homogeneous $c \in \mathbb{R}$

Now what about $A\vec{x} = \vec{b}$ if

$$\vec{b} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}? \qquad \underset{\text{particular}}{\overset{\text{can see}}{\vec{x}_p}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ works}$$

Are there other solutions?

Yes!! because we can have

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

L ch 3 4

$$A(\vec{x}_h + \vec{x}_p) = \underbrace{A\vec{x}_h}_{\vec{0}} + \underbrace{A\vec{x}_p}_{\vec{b}.}$$

(do example as well)

## The big deal

We call the subspace of $\mathbb{R}^n$ which is made up of all $\vec{x}_h$ that satisfy $A\vec{x}_h = \vec{0}$ the Nullspace of $A$
Notation $\mathcal{N}(A)$ or $N(A)$

If the $N(A) = \{\vec{0}\}$ and $\vec{b} \in C(A)$, we have a unique solution

If the $N(A)$ has dimension 1 or more & $\vec{b} \in C(A)$, we have $\infty$ly many solutions

1 lec

## How to find your nullspace

~~example:~~

~~$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \qquad \text{Solve } A\vec{x} = \vec{0}$$~~

~~$E_{21} = I$, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$~~ ~~$l_{21} = 0$, $l_{31} = 1$~~

~~$$E_{31}E_{21}A = \left[\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} b_1 \\ b_2 \\ b_3 + b_1 \end{matrix}\right]$$~~

~~← 3rd pivot is 0 this is okay!~~

# LINEAR ALGEBRA

How to find the column space and null space of a matrix $A$:

Recap: $A^{m\times n}$ $\quad \mathbb{R}^n \ \vec{x} \xrightarrow{A\vec{x}} \vec{b} \ \mathbb{R}^m$

$C(A)$ = $A$'s column space
= a subspace of $\mathbb{R}^m$
= all $\vec{b}_{m\times 1}$ that $A\vec{x}$ can reach

$N(A)$ = $A$'s nullspace
= a subspace of $\mathbb{R}^n$
= all $\vec{x}_{n\times 1}$ for which $A\vec{x} = \vec{0}$

**The method** Find $C(A)$ & $N(A)$:

$$A = \begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$m=3, n=4$

Proceed as if solving for $A\vec{x}=\vec{b}$ arbitrary $\vec{b}$.

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 10 & b_2 \\ 6 & 12 & 12 & 18 & b_3 \end{array}\right]$$

$R_2' = R_2 - \frac{2}{2}R_1 \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3' = R_3 - \frac{6}{2}R_1 \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$\sim \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 0 & 0 & 3 & 6 & b_2 - b_1 \\ 0 & 0 & 3 & 6 & b_3 - 3b_1 \end{array}\right]$$

0 pivot ok! keep going

$R_3' = R_3 - \frac{3}{3}R_2, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\sim \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 0 & 0 & 3 & 6 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 - (b_2 - b_1) \end{array}\right]$$

↖ pivot columns $\quad$ ↖ $b_3 - b_2 - 2b_1$

$x_1$ & $x_3$ are pivot variables

2nd lecture for Ch 3 $\quad$ L Ch3 5

keep going to obtain reduced row echelon form (as we did for inverses)

$R_1' = R_1 - \frac{3}{3}R_2 \quad E_{12} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\sim \left[\begin{array}{cccc|c} 2 & 4 & 0 & -2 & b_1 - (b_2 - b_1) \\ 0 & 0 & 3 & 6 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$$

Finally divide through by pivots

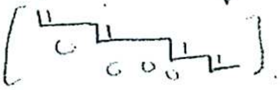
$R_1' = \frac{1}{2}R_1 \quad R_2 = \frac{1}{3}R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1 - b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2 - b_1) \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$$

$$= [R \mid \vec{d}\,]$$

pivot columns $\quad$ "free" columns

NOTES
- We can't reduce any further
- $R$ is unique for a given $A$
- row swaps are still possible
- now: pivots may appear irregularly.



- in building $R$, always introduce zeros above and below pivots.
- always divide through to change pivots into 1's.

* Definition
#pivot~~s~~ columns = rank of a matrix
We always write rank as $r$.

(Looking ahead: every matrix has an invertible square $r\times r$ matrix hiding inside it)
↑ridiculously important!!!

# LINEAR ALGEBRA

All right, so how do we find C(A) & N(A)

→ C(A): all $\vec{b}$'s for which $A\vec{x}=\vec{b}$ has a solution

3 ways at least

Our example: $\vec{d} = \begin{bmatrix} \frac{1}{2}(2b_1-b_2) \\ \frac{1}{3}(b_2-b_1) \\ b_3-b_2-2b_1 \end{bmatrix}$

[ $A\vec{x}=\vec{b}$ is solvable as long as bottom row is all 0's in $[A \mid \vec{d}]$ ]

$\Rightarrow \quad b_3 - b_2 - 2b_1 = 0$

← eq of a plane in 3d.

We could stop here, but there is a better way: columns!!

write $b_3 = b_2 + 2b_1$ , $b_2, b_1 \in \mathbb{R}$

then $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2+2b_1 \end{bmatrix} = b_1\begin{bmatrix}1\\0\\2\end{bmatrix} + b_2\begin{bmatrix}0\\1\\1\end{bmatrix}$

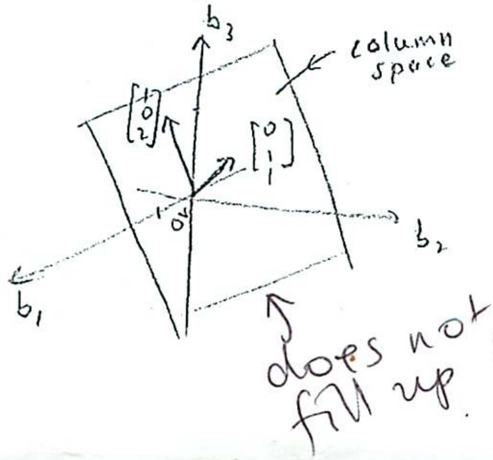
+ formal def

↑ always write this out $\qquad b_1, b_2 \in \mathbb{R}$

$C(A) = \{\vec{b} \in \mathbb{R}^3 / \vec{b} = \dots\}$

So we see C(A) is a 2-d subspace of $\mathbb{R}^m = \mathbb{R}^3$

[ Big deal $\quad r = 2 = \text{dimension of } C(A)$ (more later) ]



↑ does not fill up $\mathbb{R}^3$

n.b.

Nothing special about $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$

$\begin{bmatrix}2\\0\\4\end{bmatrix}$ & $\begin{bmatrix}0\\-1\\-1\end{bmatrix}$ would work too

also, we could have written

$b_3 - b_2 - 2b_1 = 0$

as $b_2 = b_3 - 2b_1$

then $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_3-2b_1 \\ b_3 \end{bmatrix} = b_1\begin{bmatrix}1\\-2\\0\end{bmatrix} + b_3\begin{bmatrix}0\\1\\1\end{bmatrix}$

[ There are other ways to find C(A) + More next week when we talk about bases (sec 3.5) ]

So N(A): Solve $A\vec{x} = \vec{0}$

So $b_1 = b_2 = b_3 = 0$ ← Id matrix

$\vec{0} = \vec{b}$ ↑ n.b.

$\Rightarrow$ our problem $\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\uparrow x_1 \qquad \uparrow x_3$

Here, we have a well defined procedure: express pivot variables in terms of free variables

$x_1 + 2x_2 \quad - x_4 = 0$

$\qquad\qquad x_3 + 2x_4 = 0$

pivot variables $\qquad$ free variables

$\Rightarrow x_1 = -2x_2 + x_4$

$\quad\ x_3 = -2x_4$

↙ replace pivot variables

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2+x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}1\\0\\-2\\1\end{bmatrix}$

$x_2, x_4 \in \mathbb{R}$

Linear ALGEBRA

special solutions ↓ ↘

Formally:

$$N(A) = \left\{ \vec{x} \in R^4 \;\middle|\; \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

↑ a 2-d $\overset{\text{subspace}}{\text{plane}}$ in 4-d.

Note: $n - r = 4 - 2 =$ dimension of Nullspace (more later)

n.b.

For the $A$ above, if $A\vec{x} = \vec{b}$ has a solution, there are always $\infty$ly many solutions.
Reason: $N(A)$ is not $\{\vec{0}\}$, it's a plane of vectors.

↑L9

↓L10

Let's finish this example with a $\vec{b} \in C(A)$. e.g. $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$

$$[A \mid \vec{b}] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1 - b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2 - b_1) \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Do the same thing we did to find the null space

$$x_1 + 2x_2 - x_4 = 1$$
$$x_3 + 2x_4 = 0$$

pivots

$x_1 = 1 - 2x_2 + x_4$, $x_3 = -2x_4$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 2x_2 + x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{\vec{x}_p} + \overbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}}^{\vec{x}_h}$$

$x_2, x_4 \in R$

↗ special solution

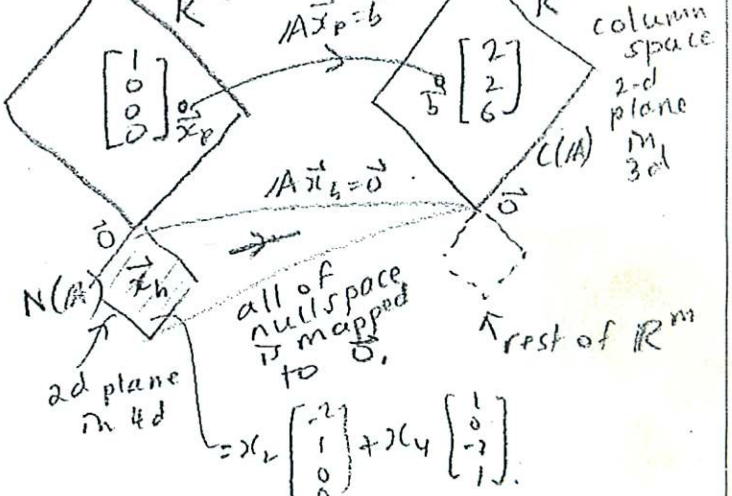
$$\vec{x} = \vec{x}_p + \vec{x}_h$$

$$A\vec{x} = \underset{\vec{b}}{\underset{\parallel}{A\vec{x}_p}} + \underset{\vec{0}.}{\underset{\parallel}{A\vec{x}_h}} = \vec{b}$$

↖ lives in null space

Notice: we can find $\vec{x}_p$ if we set $x_2, x_4 = 0$ and solve for $x_1, x_2$.

So $\vec{b}$ is reached using non-zero pivot variables & zero'd free variables.



Connection between

Form of R and special solutions

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbb{F} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \text{ for free vars}$$

$\mathbb{I}$ for pivot cols

↙ special solut.

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \qquad \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for free row variables}$$

$$-\mathbb{F} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

↙ permute $x_i$'s

most general form:

$$\mathbb{R} = \left[\begin{array}{c|c} \mathbb{I} & \mathbb{F} \\ \hline 0\,0\,0\,0\,0 & \end{array}\right] \qquad \text{special solution matrix} \quad \mathbb{N} = \left[\begin{array}{c} -\mathbb{F} \\ \hline \mathbb{I} \end{array}\right]$$

$$\mathbb{R}\mathbb{N} = \mathbb{O} = \mathbb{I}(-\mathbb{F}) + \mathbb{F}\mathbb{I}$$

We get much information from R:

① Null space
② rank r (dimension of column space)
③ and more (later)
④ free set of tofu knives

---

What can we say about $A\vec{x}=\vec{b}$ for following reduced row echelon forms of A?:

ⓐ $R=\begin{bmatrix} 1 & \infty & \infty & 0 & \infty & 0 \\ 0 & 0 & 0 & 1 & \infty & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ n.b. ∞'s are different

$\uparrow$ pivot, $\uparrow$ free $\nearrow$, $\uparrow$ pivot, $\uparrow$ free, $\uparrow$ pivot

$m=3,\ n=6,\ r=3=\#$ pivot columns

So: we immediately know that
$C(A)=R^3=R^m$ since $m=r=3$
↳ subspace of $R^m$

Null space is a $\underset{n-r}{\overset{\shortparallel}{3}}$-d subspace of $R^n=R^6$

$\Rightarrow A\vec{x}=\vec{b}$ always has a solution and there are always ∞ly many.

ⓑ $R=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $m=n=r=3$
$C(A)=R^3=R^m$

Null space is $\{\vec{0}\}$
(no free variables)

$\Rightarrow A\vec{x}=\vec{b}$ always has a unique solution $\Leftrightarrow$ A is invertible.

Observation:
If A is n×n & invertible
A's R must be the identity matrix.

ⓒ $R=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $m=4,$
$n=r=3$

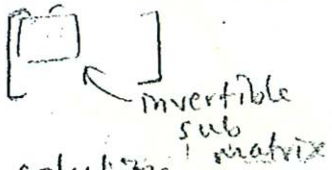
again null space is $\{\vec{0}\}$
(no free variables)
(solution to $A\vec{x}=\vec{0}$ is $\vec{x}=\vec{0}$.

but C(A) is a 3-d subspace of $R^4=R^m$

$\Rightarrow A\vec{x}=\vec{b}$ may or may not have a solution.
If it does, it is unique.

---

Summary



(i) $m=r,\ n=r$
$\overset{\uparrow}{A}\vec{x}=\vec{b}$ has 1 solution always
square & invertible

(ii) $m\overset{\shortparallel}{=}r,\ n>r$ (wide [[ ]])
$A\vec{x}=\vec{b}$ has ∞ly many solutions always

(iii) $m>r,\ n=r$ tall [| |]
$A\vec{x}=\vec{b}$ has 0 or 1 solution
(no Null space)

(iv) $m>r,\ n>r$
$A\vec{x}=\vec{b}$ has 0 or ∞ly many solutions

(Add qualities of N(A) C(A)
Make a chart)!

# LINEAR ALGEBRA

## Section 3.5

+ finding column space on $A^T$ + rowspace

Ch.3 L 9

Today:

[ Bases and spanning sets & dimensions of subspaces ]

+ [ Let's explore $A\vec{x}=\vec{b}$ when $A=\begin{bmatrix}1&2&1\\2&4&2\end{bmatrix}$

First: find the column space & null space of $A$, $C(A)$ & $N(A)$

Solve $\left[\begin{array}{ccc|c}1&2&1&b_1\\2&4&2&b_2\end{array}\right]$ by row reduction
$(A|\vec{b})$

$R_2' = R_2 - \frac{2}{1}R_1$

$$\sim \left[\begin{array}{ccc|c}1&2&1&b_1\\0&0&0&b_2-2b_1\end{array}\right]$$

pivot col   free cols

Column space: $b_2-2b_1=0 \Rightarrow b_2=2b_1$

n.b. obvious from $A$!

$$\vec{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}=\begin{bmatrix}b_1\\2b_1\end{bmatrix}=b_1\begin{bmatrix}1\\2\end{bmatrix} \quad b_1\in R$$

[ Null space: $\vec{b}=\begin{bmatrix}0\\0\end{bmatrix}$

$x_1+2x_2+x_3=0$

$$\Rightarrow \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2x_2-x_3\\x_2\\x_3\end{bmatrix}$$

$$=x_2\begin{bmatrix}-2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix} \quad x_2,x_3\in R$$

↑↗ free variable

Lec 11

Observe: column space and nullspace are subspaces of $R^m$ & $R^n$

$C(A)\subset R^m \qquad N(A)\subset R^n$

Take $N(A)$:

$N(A)$ comprises all linear combinations of $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Check Subspace properties

① if $\vec{x},\vec{y}\in N(A)$, $\vec{x}+\vec{y}\in N(A)$

$$\vec{x}=c_1\begin{bmatrix}-2\\1\\0\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix},\quad \vec{y}=c_3\begin{bmatrix}-2\\1\\0\end{bmatrix}+c_4\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$\Rightarrow \vec{x}+\vec{y}=(c_1+c_3)\begin{bmatrix}-2\\1\\0\end{bmatrix}+(c_2+c_4)\begin{bmatrix}-1\\0\\1\end{bmatrix}\in N(A)$$

② $\vec{x}\in N(A)\Rightarrow c\vec{x}\in N(A)$ clear

③ $\vec{0}\in N(A)$? Sure: $\vec{0}=0\begin{bmatrix}-2\\1\\0\end{bmatrix}+0[$

In general, sets made up of all linear combinations of a collection of vectors are subspaces.

Words: we say $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ "span" the nullspace of $A$.

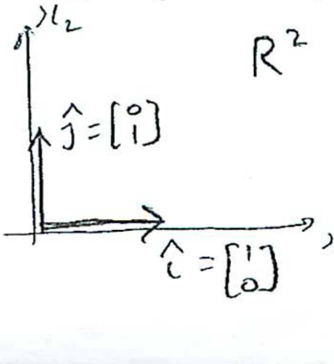
and that $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ is a spanning set of $N(A)$

Note well: other sets could span $N(A)$... (in fact ∞ly many)

$\left\{2\begin{bmatrix}-2\\1\\0\end{bmatrix},3\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}+\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right.$

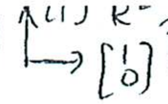
More examples: Spanning sets for $R^2$:

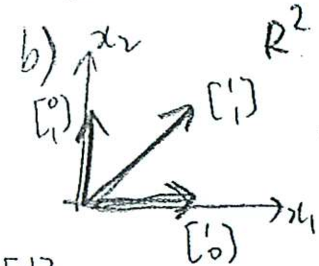
a)



$\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$ span $R^2$

B LINEAR ALGEBRA

a) 

b) 

$\mathbb{R}^2$ $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

span $\mathbb{R}^2$

c) 

← spans $\mathbb{R}^2$

Observe a) & c) examples are special because we need both vectors

for b) we could take any one vector away, and the set of vecs would span $\mathbb{R}^2$ still. (overkill redundancy)

Words: a) & c) examples are a linearly independent set

b) example is a linearly dependent set

Defn: a set of vectors $\{\vec{a}_1, \vec{a}_2, .., \vec{a}_n\}$ is linearly independent if

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = 0$$

is solved only by $x_1 = x_2 = \ldots = x_n = 0$

↑ homogeneous equation!!

Why?

If some of the $x_i \neq 0$, we can express one vector in terms of the others

expand!

eg. b) above $x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} + x_3\begin{bmatrix}1\\1\end{bmatrix} = 0$

is solved by $x_1 = 1$
$x_2 = 1$
$x_3 = -1$

and clearly

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}.$$

Defn A spanning set that is linearly independent is called a basis

Plural: Bases
bay-seez

Ch 3
L 10

(or a minimal spanning set)

Note Bases are not unique but some are better than others...

↑ more later.

It's good to have a basis: because then we have a unique representation of each point in our subspace.

e.g. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$: $\begin{bmatrix}33\\72\end{bmatrix} = 33\begin{bmatrix}1\\0\end{bmatrix} + 72\begin{bmatrix}0\\1\end{bmatrix}$

weird: everything is in terms of {}

only way

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$: $\begin{bmatrix}33\\72\end{bmatrix} = 33\begin{bmatrix}1\\0\end{bmatrix} + 72\begin{bmatrix}0\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix}$

$= 0\begin{bmatrix}1\\0\end{bmatrix} + 39\begin{bmatrix}0\\1\end{bmatrix} + 33\begin{bmatrix}1\\1\end{bmatrix}$

not unique

(n.b. $\infty$ly many ways to represent a point $\iff$ exactly the same as $A\vec{x} = \vec{b}$ having $\infty$ly many solutions $\iff N(A) \neq \{\vec{0}\}$.)

Back to our $N(A)$ & $C(A)$

$\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ is ~~a~~ basis for $N(A)$

$\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ is a basis for $C(A)$

next page

Defn: ↓ big deal!!!

The dimension of a space is the # vectors in any basis for that space

# LINEAR ALGEBRA

So for $\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$ $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\dim C(\mathbb{A}) = 1$

$\dim N(\mathbb{A}) = 2$

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Now we are getting somewhere...

In general, $\boxed{\begin{aligned} \dim C(\mathbb{A}) &= r = \text{the rank of } \mathbb{A} \\ &= \#\text{pivots} \\ \dim N(\mathbb{A}) &= n - r \end{aligned}}$

v. important!!

Why?

First $N(\mathbb{A})$:

recall for our example $\vec{x}_h = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ ← basis vectors

why indep

↑ our free variables are $x_2$ & $x_3$

$\Rightarrow$ Number of basis vectors

$\equiv$ # of free variables

$\equiv n - r$

↑ total # of variables ↑ # of pivot variables

$C(\mathbb{A})$: key point ①— when we do row operations, the relationships between columns do not change (concentrate!)

e.g. $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}$ clearly col 1 = col 2 + col 3

now do some row ops: $R_2' = R_2 - 2R_1$
$R_3' = R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix}$ still have col 1 = col 2 + col 3

key point 2:

Ch3 L 11

In $\mathbb{R}$, the reduced row echelon form of $\mathbb{A}$, we can see that free cols can easily be made by combinations of pivot cols.

e.g. (from prev lec)

$\begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\mathbb{A}$ $\mathbb{R}$

pivot

col 2 = 2×col 1 ⇐ see from $\mathbb{R}$
cos 4 = −1×col 1 + 2×col 3

Observe same relations hold for $\mathbb{A}$ !!

- Be careful; relationships between columns are not changed by row reduction but cols in $\mathbb{R}$ are not in column space!!!
- Row reduction alters columns, but $N(\mathbb{R}) = N(\mathbb{A})$ $C(\mathbb{A}) \neq C(\mathbb{R})$

What this means: we see the pivot columns in $\mathbb{A}$ form (as found in $\mathbb{R}$) a basis for $C(\mathbb{A})$ & there are exactly r of them. (by defn of r.)

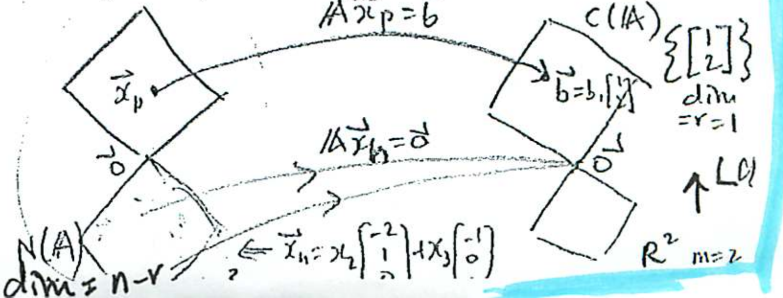
Our small example of the day

$\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$ $\mathbb{R} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

basis vector ← pivot col

$\dim C(\mathbb{A}) = r = 1$

The big pic so far for $\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$:



# LINEAR ALGEBRA L12

Two more subspaces to go:

First: row space

= subspace of $R^n$ ← where $\vec{x}$ lives

spanned by row vectors of $A$.

e.g. $\begin{bmatrix} 1 & -4 & 1 \\ 2 & 7 & 2 \\ 3 & 2 & 3 \end{bmatrix}$

Recall: Column space is where $\vec{b}$ lives if $A\vec{x}=\vec{b}$ is solvable

$C(A) \subset R^m$ whereas row space of $A \subset R^n$

cf. the right stuff Braveheart this lecture has nothing to do with them

Seems like a fine space but is it useful like column space? Yes!

~~Three~~ ~~Four~~ Five!

Big deals about Row Space:

**BD#1** If $\vec{x} \in A$'s row space (→ can be expressed as a combination of row vectors)

$A\vec{x} \neq \vec{0}$ unless $\vec{x}=\vec{0}$.

our example $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$ Take $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ← clearly in row space

$A\vec{x} = \begin{bmatrix} 6 \\ 12 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← our basis vector for col space

what is row space for $A$?

by inspection

row space $= \left\{ \vec{x} \in R^3 \;\middle|\; \vec{x} = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ $c \in R$

So $\vec{x}_p$ must (somehow/kinda) live in row space!!

($\vec{x}_h$ lives in nullspace)

$\vec{x} = \vec{x}_p + \vec{x}_h$

$A\vec{x} = A\vec{x}_p + \underbrace{A\vec{x}_h}_{\vec{0}}$

$\overset{\|}{\vec{b}}$ ← our particular solution is some combination of $A$'s rows

**BD#2** Any $\vec{x}$ in $A$'s row space is orthogonal ($\perp$) to any $\vec{x}$ in $A$'s null space.

**Ch3 L12**

(or row space & null space are at "right angles" to each other)

Reason: By definition, if $\vec{x} \in N(A)$ then $A\vec{x}=\vec{0}$ so the dot product of all the rows of $A$ with $\vec{x}$ must be 0.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \left( c_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= c_1\begin{bmatrix} 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ✓$$

4 $A^T$ 5 $N(A)$ [illegible] $= R^n$

**BD#3** The row space of $A$ is the same as the row space of $\mathbb{R}$!

$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}} \rightsquigarrow \underset{\mathbb{R}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$ so using row reductions we find a basis for $A$!

(2, 3) (2, 6) (6, 12)

$$\begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right.$$

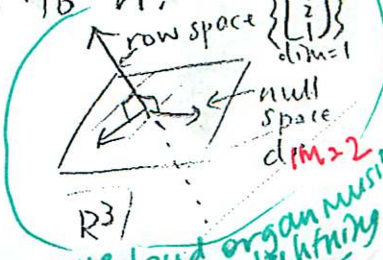
[# non-zero rows in $\mathbb{R} = r$, the rank]

$\Rightarrow$ dim ($A$'s row space) $= r$ = # basis vectors for $A$'s row space !!!

same as $C(A)$

**BD#4** Dimensions of $A$'s null space $(=n-r)$ & $A$'s row space $(=r)$ add up to $n$.

row & null space neatly divide up $R^n$ ← where x lives



cue loud organ music + lightning

BD$\frac{4}{5}$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$

We see $(A^T)$'s column space is $A$'s row space!! huge!

Plus $A$'s column space is $(A^T)$'s row space!!

[So we'll write the row space of $A$ as $C(A^T)$

---

Bonus big deal:

$A^T$ has a null space too!! ~~Its~~ dimension must be $m-r$!

[We'll call $N(A^T)$ the left null space of $A$.

Reason: if $\vec{y} \in N(A^T) \subset R^m$ then by defn $A^T\vec{y} = \vec{0}$

Take transpose of each side:

$$(A^T\vec{y})^T = \vec{0}^T$$

$$\overset{\shortparallel}{\vec{y}}^T (A^T)^T = \vec{0}^T$$

$$\vec{y}^T A = \vec{0}^T \leftarrow \text{row vec}$$

row vec ↗ so $\vec{y}^T$ is a 'left' null vector for $A$ b/c it multiplies on the left

+ Nullspace

‖

Right Null space.

Ch 3 L 13

What is $N(A^T)$?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

do row ops

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} \overset{\text{pivot}}{1} & \overset{\text{free}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \checkmark \\ \checkmark \end{array}$$

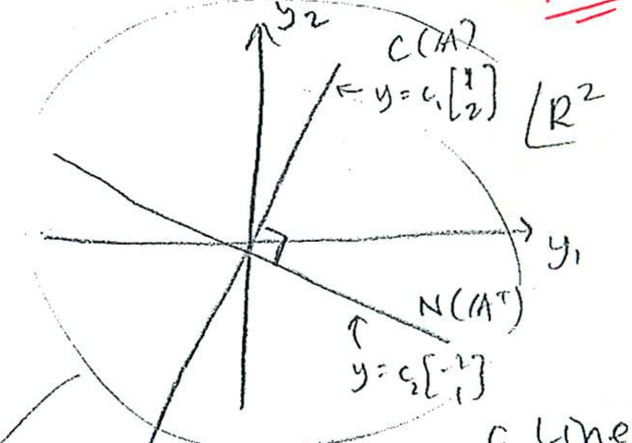
$\Rightarrow y_1 + 2y_2 = 0 \Rightarrow y_1 = -2y_2$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2y_2 \\ y_2 \end{bmatrix} = y_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad y_2 \in R$$

check $C(\overset{\text{// row space of } A^T}{A})$ vecs are ⊥ to $N(A^T)$ vecs ✓.

$$[1\ 2]\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0.$$

+ show leftness



[Fundamental Theorem of Linear Algebra part 1

$\dim C(A) = r$

$\dim N(A^T) = m - r$

& $\dim C(A) + \dim N(A^T) = r + (m-r)$
$= m.$

Similarly

$\dim C(A^T) + \dim N(A) = r + (n-$
$= n$

~~discuss direct sums of subspaces~~

~~[All vectors in $R^2$ ($R^m$) can be expressed as a linear combination of vectors in $N(A^T)$ & $C(A)$~~

~~similarly for $N(A)$ & $C(A^T)$~~

later?

~~Write: $R^m = N(A^T) \oplus C(A)$~~

~~$R^n = N(A) \oplus C(A^T)$~~

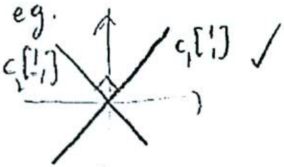
We need
Some definitions to go further

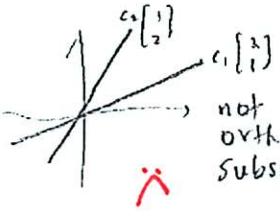
1) If $\vec{x} \cdot \vec{y} = 0$ we say say $\vec{x}$ & $\vec{y}$ are orthogonal

[in a sense $\vec{x}$ contains none of $\vec{y}$ and vice versa

② We say two subspaces V & W are orthogonal if all vectors in V are orthogonal to all vectors in W

eg.





not orthog. subspaces

Now we showed that all vectors in C(A) are ⊥ all vectors in N(A^T)

Similarly C(A^T) ⊥ N(A) (use words)

But wait, there's more!
Because their dimensions add up to the dimensions of their spaces, we say these subspaces are orthogonal complements of each other *

$\vec{S}$, $S^{\perp}$ ← perp.

Fundamental Theorem part 2:

[Null space N(A) is the orthogonal complement of row space C(A^T)

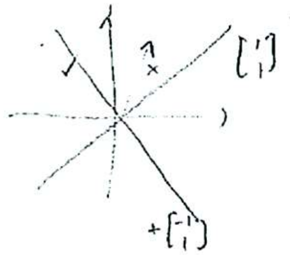
Left Nullspace N(A^T) is the orthogonal complement of column space C(A)]

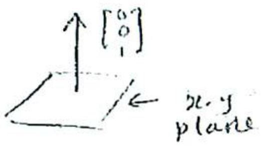
← write out full theorem

Ch 4
L1.

* defn.
orthogonal complement of $S \subset V$

$$\pm S^{\perp} = \{ \vec{x} \in V \mid \vec{x} \cdot \vec{s} = 0 \ \forall \vec{s} \in S \}.$$





Last

The bases of C(A^T) & N(A) combine to give a basis of $R^n$

The bases of C(A) & N(A^T) combine to give a basis of $R^m$.

LINEAR ALGEBRA

L13

Big picture for the 4 kinds of A

\# solns

m=n=r

1

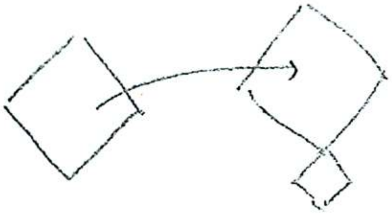
m=r
n>r

wide

∞

m>r
n=r

tall



1 or ∞

m>r
n>r

shape unknow

0 or ∞

non trivial null space ⇒ ∞ possibity

non trivial left null space ⇒ 0 possibility

+dimensions

Ch4 L3

| row $C(A^T)$ | null $N(A)$ | left null $N(A^T)$ | col $C(A)$ |
|---|---|---|---|
| $\mathbb{R}^n$ | $\{\vec{0}\}$ | $\{\vec{0}\}$ | $\mathbb{R}^m$ |
| | | $\{\vec{0}\}$ | |
| | $\{\vec{0}\}$ | | |
| | | | etc. |

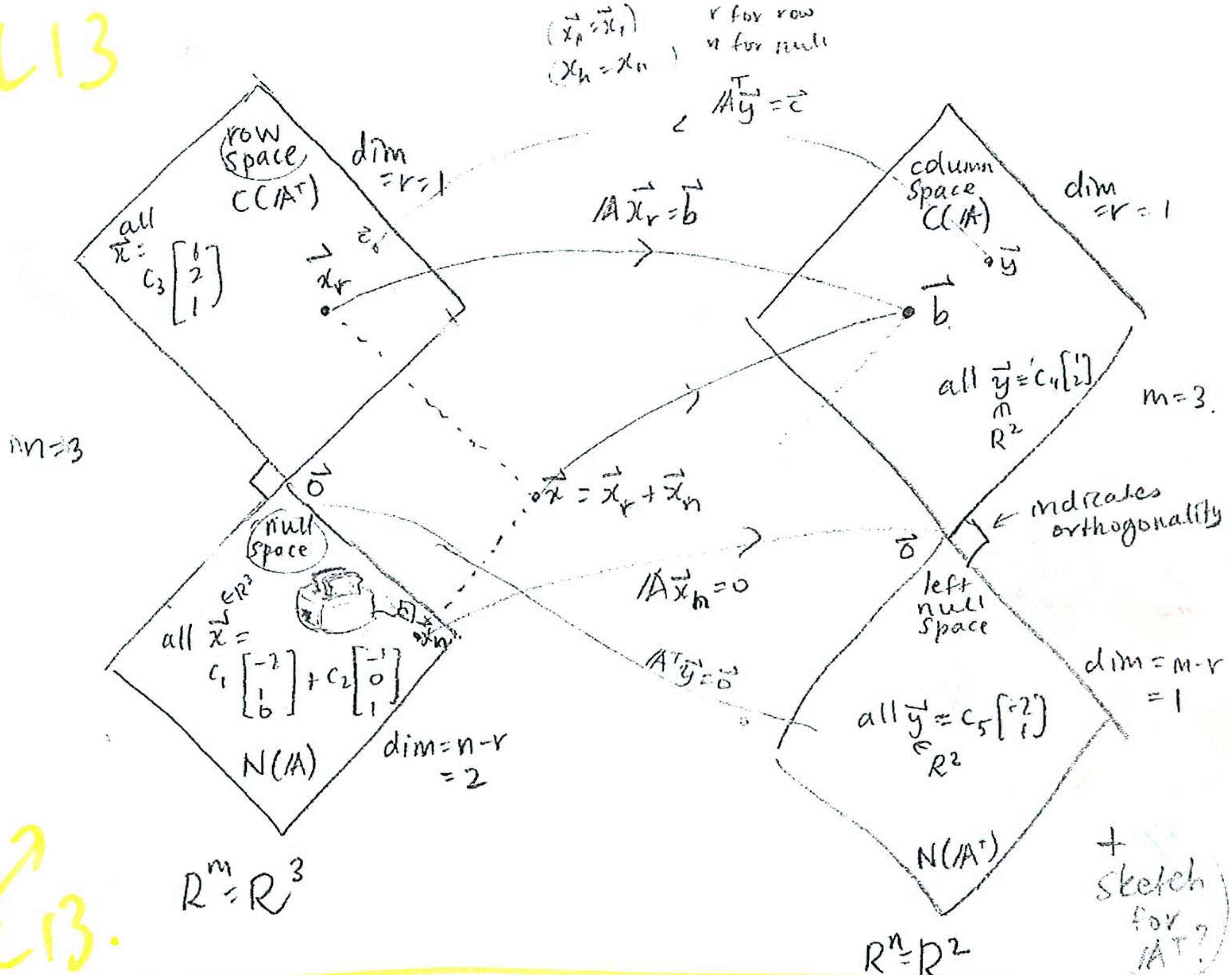
$+\mathbb{R}$

LINEAR ALGEBRA

Ch4
L2

# The Big Picture for $A\vec{x}=\vec{b}$ when $A=\begin{bmatrix}1&2&1\\2&4&2\end{bmatrix}$

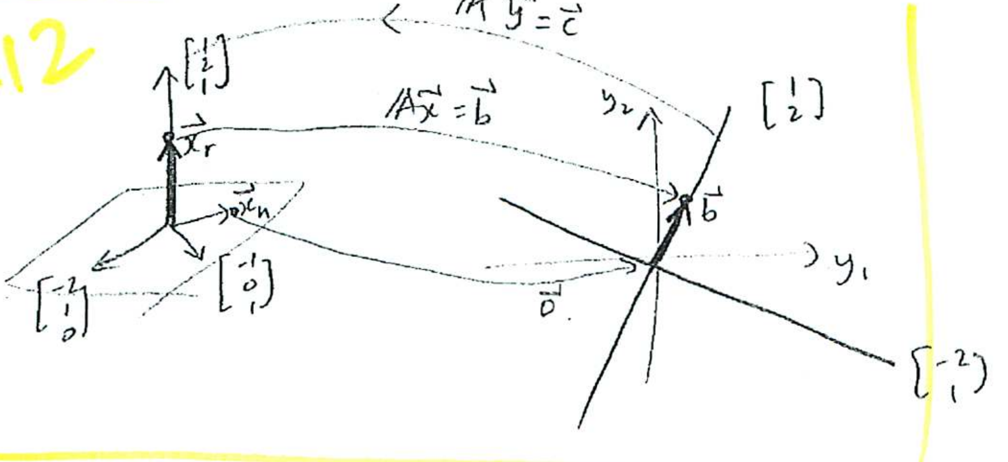
L13



+ sketch for $A^T$?

L13.

L12



$A$ & $A^T$
map between
$r=1$ dim
subspaces
of $R^n$ & $R^m$

LINEAR ALGEBRA

* One more thing about bases for our four subspaces:

(1st way)

[ We found C(A) by solving $\mathbb{A}\vec{x}=\vec{b}$ for a general $\vec{b}$ and the finding which $\vec{b}$'s were reachable.

A nicer plan: (2nd way)

The row space of $\mathbb{A}^T$ is the same as C(A)

⇒ Find the RREF of $\mathbb{A}^T$ and read off the basis vectors.

(Easy & gives a unique basis) since $\mathbb{R}$ is unique

e.g. ① $\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

$$\mathbb{A}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{reduce}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

⇒ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the basis vector

②

$$\mathbb{A} = \begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix}$$

$$\mathbb{A}^T = \begin{bmatrix} 2 & 2 & 6 \\ 4 & 4 & 12 \\ 3 & 6 & 12 \\ 4 & 10 & 18 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 6 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for C(A): $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

excellent...

Ch4 L4

[ Projections

(moving towards handling $\mathbb{A}\vec{x}=\vec{b}$ when $\vec{b} \notin C(\mathbb{A})$) ]

(3rd way) (really doing this for understanding)

*

Recall $\mathbb{A} = \begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix}$

$$\Rightarrow \mathbb{R} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

① We observed that row reduction does not change relationships between columns

done

② $\mathbb{R}$ shows us that the free columns can be formed by the pivot columns

$$\mathbb{R} \Rightarrow \underset{c_2}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}} = 2\underset{c_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \quad \& \quad \underset{c_4}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} = -\underset{c_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + 2\underset{c_3}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$\mathbb{A} \Rightarrow \begin{bmatrix} 4 \\ 4 \\ 12 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \quad \& \quad \begin{bmatrix} 4 \\ 10 \\ 18 \end{bmatrix} = -\begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}$$

Pivot columns in $\mathbb{A}$ (not $\mathbb{R}$!) form a basis for C(A)

↘ but need $\mathbb{R}$ to find them!

Basis is $\left\{ \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix} \right\}$

[ Alert: C(A) & C(R) are different! but A's & R's column relationships are the same ]

[ Note: bases obtained may well be different using different methods!

We expect this... there are only many bases for a given space

Thursday 10/16

2$^{00}$ Thursday / 111 Lafayette ①

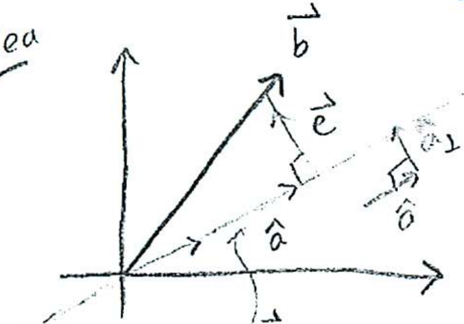
LINEAR ALGEBRA

Section 4.2

## Projections

We are Moving towards handling $\mathbb{A}\vec{x}=\vec{b}$) when there are no solutions

idea



[given $\vec{b}$, break it into components in direction of $\hat{a}$ & $\perp$ to $\hat{a}$

improve..

write

$$\vec{b}=\vec{p}+\vec{e}$$

$\vec{p}$: projected component

$\vec{e}$: error

Reason [In solving $\mathbb{A}\vec{x}=\vec{b}$, if $\vec{b}\notin C(\mathbb{A})$ we can still solve $\mathbb{A}\vec{x}^*=\vec{p}$. to obtain our best approximation (left nullspace will matter here!)

To find $\vec{p}$ & $\vec{e}$:

We want $\vec{p}\parallel\vec{a}$ & $\vec{e}\perp\vec{a}$

Mathematically: $\vec{a}\cdot\vec{e}=0$ or $\vec{a}^T\vec{e}=0$ & $\vec{p}=\gamma\vec{a}$

$$\vec{a}^T\vec{b}=\vec{a}^T(\vec{p}+\vec{e})$$

$$\Rightarrow \vec{a}^T\vec{b}=\vec{a}^T\gamma\vec{a}+\cancel{\vec{a}^T\vec{e}}^{\,0}$$

$$\Rightarrow \gamma=\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} \qquad \Rightarrow \boxed{\vec{p}=\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\vec{a}}$$

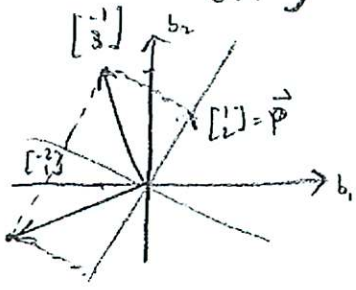
Example: project

$$\vec{b}=\begin{bmatrix}-3\\-1\end{bmatrix} \text{ onto } \begin{bmatrix}1\\2\end{bmatrix}=\vec{a}$$

$$\gamma=\frac{-5}{5}=-1 \Rightarrow \vec{p}=-1\cdot\vec{a}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$\vec{e}=\vec{b}-\vec{p}=\begin{bmatrix}-3\\-1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\+1\end{bmatrix}$$

Ch4
L5



show $\vec{e}\perp\vec{p}$

Something sneaky:

$$\vec{p}=\left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\underset{n\times1}{\vec{a}}=\vec{a}\left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)$$

1×1

Switcharoo

$$=\left(\frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}\right)\vec{b},$$

contains only $\vec{a}$

projection matrix $=\mathbb{P}$ $\underline{n\times n}$

$$\vec{a}=\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow \frac{1}{5}\begin{bmatrix}1&2\\2&4\end{bmatrix}=\mathbb{A}_{proj}$$

note $\mathbb{A}_{proj}=\mathbb{A}_{proj}^T$

$$\mathbb{A}_{proj}\vec{b}=\frac{1}{5}\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}-3\\-1\end{bmatrix}$$

$$\vec{p}=\frac{1}{5}\begin{bmatrix}-5\\-10\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

as before.

$$\vec{e}=(\mathbb{I}-\mathbb{A}_{proj})\vec{b}=\begin{bmatrix}-2\\1\end{bmatrix}$$

Next: project onto column space (or any subspace)

⊕ show $\vec{p}=\hat{a}\hat{a}^T\vec{b}$

Collect assignment #5.

→ 44

L14

LINEAR ALGEBRA

L14

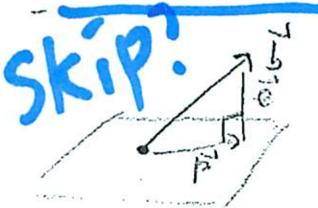
Projecting a vector $\vec b$ onto a subspace $S \subset \mathbb{R}^m$

① How to do

② Example

③ Main use (solving $A\vec x = \vec b$ when we can't solve $A\vec x = \vec b$)

Skip?



← break $\vec b$ into two pieces

$\vec p$ = projection which lives in $S$

$\vec e$ = error which lives in $S^\perp$

subspace of $R^m$ $= S$

We must have a description of $S$, in particular a basis: $\{\vec a_1, \vec a_2, \dots, \vec a_r\}$ $r \le m$

We want these two things to be true:

(a) $\vec p = \vec a_1 x_1^* + \vec a_2 x_2^* + \dots + \vec a_r x_r^*$ ← $\vec p$ lives in $S$

$$= \begin{bmatrix} | & | & & | \\ \vec a_1 & \vec a_2 & \dots & \vec a_r \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1^* \\ \vdots \\ x_r^* \end{bmatrix} = \underset{m\times r}{A}\,\underset{r\times 1}{\vec x^*}$$

(b) $\vec e$ must be $\perp$ all $\vec a_i$

$$[\,-\ \vec a_1\ -\,]\begin{bmatrix} | \\ \vec e \\ | \end{bmatrix} \quad [\,-\ \vec a_2\ -\,]\begin{bmatrix} | \\ \vec e \\ | \end{bmatrix} \quad \vdots$$

$\Rightarrow$ $\vec a_1^T\vec e = 0$

$\vec a_2^T\vec e = 0$

$\vdots$

$\Rightarrow$ $A^T\vec e = \vec 0$

$\Downarrow$

$A^T(\vec b - \vec p) = \vec 0$

$\Downarrow$

$A^T\vec b = A^T\vec p$

combine (a) & (b)

piece 1/3

$$\boxed{\underset{r\times m}{A^T}\,\underset{m\times 1}{\vec b} = \underset{r\times m}{A^T}\,\underset{m\times r}{A}\,\underset{r\times 1}{\vec x^*}}$$

$\vec p$ ← called the normal equation

very important!

Next: see that $A^TA$ is square and invertible ← reason later

fold into $A\vec x = \vec b$

Ch4 L6

$\Rightarrow$ $\vec x^* = (A^TA)^{-1}A^T\vec b$

piece 2/3

$\Rightarrow$ $\boxed{\vec p = A(A^TA)^{-1}A^T\vec b}$

since $\vec p = A\vec x^*$

And last, we can identify a matrix that does the projection for us:

$\vec p = \mathbb{P}\vec b$ where

piece 3/3

$\mathbb{P} = A(A^TA)^{-1}A^T$

later. L15

Warning!

$(A^TA)^{-1} \underset{\text{not nec}}{\neq} A^{-1}(A^T)^{-1}$

$A$ may not be square!

$A^{-1}$ may not exist if $A$ is square!

↳ okay if columns are a basis.

but $A^TA$ is always square $(r\times r)$

$\boxed{A^TA \text{ is invertible if } A\text{'s columns are linearly independent}}$

$\Leftrightarrow$ $A\vec x = \vec 0$ only has $\vec x = 0$ as a solution

$\Leftrightarrow$ $A$'s nullspace is $\{\vec 0\}$

Show $A^TA$ & $A$ have the same nullspace

(if $\vec x \in N(A)$ then $\vec x \in N(A^TA)$) and vice versa

"⇒" if $A\vec x = \vec 0$ then $A^T(A\vec x) = A^T\vec 0 = \vec 0$ ✓

"⇐" if $(A^TA)\vec x = \vec 0$ then $\vec x^TA^TA\vec x = \vec x^T\vec 0$

(?)

$= (A\vec x)^TA\vec x$ $\vec 0$

$= \|A\vec x\|^2 = 0$

So $A\vec x$ must be $\vec 0$

# LINEAR ALGEBRA

$$\begin{matrix} 1 & 1 & -1 & 2 \\ 7 \, 1 & 0 & 11 & : 3 \\ ? & -1 & 0, & 1 \end{matrix}$$

Example:

Project $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ onto subspace of $\mathbb{R}^3$ with basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

plane $x_1 + x_2 + x_3 = 0$.

?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$ ← guaranteed symmetric & invertible

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbb{P} = A (A^T A)^{-1} A^T$$

n.b. $\underline{\mathbb{P} = \mathbb{P}^T}$

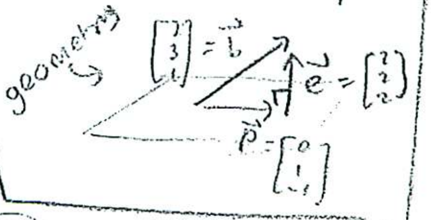
$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -7 & -1 & 2 \end{bmatrix}$$

$$\vec{p} = \mathbb{P}\vec{b} = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \left( = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$$

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$ ← check $\vec{e} \perp \vec{p}$ ✓

↑



onus: we could use calculus...

$$F(\vec{x}) = \| A\vec{x}^* - \vec{b} \|^2$$ ← could use any power (even)

minimize f w.r.t. $\vec{x}$

lifferentiate, set to 0 etc.

- equivalent
- linear algebra is more fun, a better story

origin of "least squares solution"

Ch4
L7

And why are we doing all this?
For the love of $A\vec{x} = \vec{b}$...

$\boxed{I \heartsuit A\vec{x} = \vec{b}}$

An excellent use of projection: dealing with $A\vec{x} = \vec{b}$ when there is no solution.

( Aim: if $\nexists \, \vec{x}$ s.t. $A\vec{x} = \vec{b}$, we find $\vec{x}^*$ such that $A\vec{x}$ is as close to $\vec{b}$ as possible )

The Linear Algebra way:
Take $\vec{b}$ and break it into pieces: $\vec{p} \in C(A)$ column space
$\vec{e} \in N(A^T)$ left null space

all good
We know we can reach $\vec{p}$ and that we cannot reach $\vec{e}$

Proceed as before — project $\vec{b}$ onto column space.

One crucial difference: <u>before</u> we had a basis for our subspace and we formed $A$ using this basis.

<u>Now</u>: we have $A$ and its column vectors span $C(A)$ BUT may not be a basis.

We could find a basis for $C(A)$ but let's see what being lazy gets us.

Okay, so, imagine $A\vec{x} = \vec{b}$ won't work.

We solve $A\vec{x}^* = \vec{p}$ instead

To find $\vec{p}$ & $\vec{e}$:

*

# LINEAR ALGEBRA

$\vec{e} \perp$ to all columns in $\mathbb{A}$

$$\mathbb{A}^T \vec{e} = \vec{0} \qquad (\vec{e} \in N(\mathbb{A}^T))$$

$$\Rightarrow \mathbb{A}^T(\vec{b} - \vec{p}) = \vec{0}$$

$$\Rightarrow \mathbb{A}^T(\vec{b} - \mathbb{A}\vec{x}^*) = \vec{0}$$

$$\Rightarrow \mathbb{A}^T\vec{b} - \mathbb{A}^T\mathbb{A}\vec{x}^* = \vec{0}$$

$$\Rightarrow \underbrace{\mathbb{A}^T\mathbb{A}}_{\text{new } \mathbb{A}} \vec{x}^* = \underbrace{\mathbb{A}^T\vec{b}}_{\text{new } \vec{b}}$$

← burst into flames

Same story

Normal Equation

$$\mathbb{A}'\vec{x}^* = \vec{b}'$$

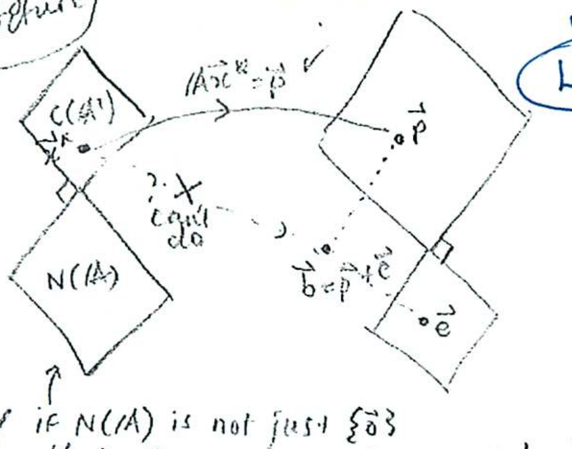
BUT

This is as far as we go because $\mathbb{A}^T\mathbb{A}$ may not be invertible (it is square but $\mathbb{A}$'s columns may be dependent),

✓ And $\mathbb{A}'\vec{x}^* = \vec{b}'$ is easy to solve using $\mathbb{L}\,\mathbb{U}$ decomp.

LIS

big picture



(if $N(\mathbb{A})$ is not just $\{\vec{0}\}$ then there are $\infty$ly many solns)

V. Important Example:

fit a straight line to a set of data points

fundamental scientific exercise

want to get this far.. →

*

Q1 4.3

Ch 4 L8

$b_t = 0, 8, 8, 20$

at times $t = 0, 1, 3, 4$

what is best straight line fit?

$$p_t = C + Dt \qquad \begin{pmatrix} C + D.0 = 0, & C + D.3 = 8 \\ C + D.1 = 8, & C + D.3 = 20 \end{pmatrix}$$

matrix form (?)

$$\underset{\mathbb{A}}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ t^0 & t^1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}}$$

clear $\vec{b}$ is not in $\mathbb{A}$'s column space.

Solve $\mathbb{A}^T\mathbb{A}\vec{x}^* = \mathbb{A}^T\vec{b}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \Rightarrow \vec{x}^* = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

↑ symmetric? ✓

$C = 1, D = 4$

$$p_t = 1 + 4t.$$

$p_0 = 1, p_1 = 5, p_3 = 13, p_4 = 17.$

error:

$$\underset{\vec{e}}{\vec{b} - \vec{p}} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$$

$$\|\vec{e}\|^2 = 44.$$

~~Show~~ $t^2$

So to repeat: if $\mathbb{A}\vec{x} = \vec{b}$ cannot be solved (not a big enough column space) we can always solve $\mathbb{A}^T\mathbb{A}\vec{x}^* = \vec{b}$ to find the "least squares solution" or best approximation..

Extra notes: if $\mathbb{A}$ is $m \times n$ then normal equation will be $n \times n$ (in above example $4 \times 2 \rightarrow 2 \times 2$ problem)

$10^6 \times 2 \rightarrow 2 \times 2$ "

can fit any polynomial

eg. $p_t = C + Dt + Et^2 + Ft^3$ ✓

$m = \#$ datapoints

$n =$ order of polynomial $+ 1$

# LINEAR ALGEBRA

Ch4
L9

section 4.4 Today

[Creating happy bases (base-eez)

1] • orthonormal & orthogonal bases
2] • the Gram-Schmidt process
3] • What this all means for $A\vec{x}=\vec{b}$

[next up: eigenvectors, eigenvalues, & determinants]

=

1] We've been finding bases for our four fundamental Subspaces. They just pop out and exact nature depends on method used.

There is a better way... Why?

Orthogonality make a basis a happy basis

aside (like  story orthog of subspaces)

ex 1

$$\left\{ \overset{\vec{a}_1}{\begin{bmatrix}-2\\0\\1\end{bmatrix}}, \overset{\vec{a}_2}{\begin{bmatrix}1\\1\\2\end{bmatrix}} \right\}$$

basis for a plane in $R^3$

$$\vec{a}_1^T\vec{a}_2 = \underset{\text{dot product}}{[-2\ 0\ 1]\begin{bmatrix}1\\1\\2\end{bmatrix}} = -2+0+2=0$$

(say basis is orthogonal) ✓

ex 2

$$\left\{ \overset{\vec{a}_1}{\begin{bmatrix}1\\3\\7\end{bmatrix}}, \overset{\vec{a}_2}{\begin{bmatrix}1\\1\\2\end{bmatrix}} \right\}$$ No!

$$\begin{bmatrix}1\\3\\7\end{bmatrix} = \underbrace{\begin{bmatrix}-2\\0\\1\end{bmatrix}+3\begin{bmatrix}1\\1\\2\end{bmatrix}}_{\vec{a}_1 \text{ has a piece of } \vec{a}_2} \leftarrow$$

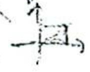
(can you see projections may be useful here?)

Important:
if we could remove this piece of $\vec{a}_2$ (the $3\begin{bmatrix}1\\1\\2\end{bmatrix}$) from $\vec{a}_1$, then we could create an orthogonal basis

=

All right, why do we want a orthogonal basis?

[Main reason: Representation of Vectors is very clean. Information contained in each basis vector is unique]

$3\hat{i}+2\hat{j}$ 

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(Bonus: An orthogonal set of vectors is automatically linearly independent (and ∴ must form a basis for the subspace they span)

**Reason** given $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$

$\vec{v}_i \perp \vec{v}_j \ \forall\ i,j \quad i \neq j$

[Assume linear dep.
$c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$
some $c_i$'s $\neq 0$.]

one at a time ↙ $* \vec{v}_i^T x$

What is the length of each side?

RHS $\vec{0}^T\vec{0}=0$

LHS $(\quad)(\quad) = \vec{c}_1^{\,2}\overset{>0}{\|\vec{v}_1\|^2} + \dots + c_k^2\overset{>0}{\|\vec{v}_k\|^2}$

only way LHS $=0$ is if all $c_i=0$. Contradiction...

=

Extra happy Bases:
[orthonormal basis]

Orthogonal + all vectors are of unit length (of length 1)

# LINEAR ALGEBRA

Ch 4
L 10

ex. In $R^3$ $\{\hat{i}, \hat{j}, \hat{k}\}$ = natural basis

orthonormal.

Observe: Easy to create an orthonormal basis if you have an orthogonal basis:

[Just divide each vector by its length]

ex. $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \to \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

* → first!!!

Next if we have a matrix $\underset{m \times n}{A}$ whose columns form an orthonormal basis for column space (in $R^m$) then

$$\underset{n \times m}{A^T} \underset{m \times n}{A} = \underset{n \times n}{I}$$

$$n \underbrace{\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \end{bmatrix}}_{m} \; m \underbrace{\begin{bmatrix} | & | & \\ \vec{a}_1 & \vec{a}_2 & \cdots \\ | & | & \end{bmatrix}}_{n} = n \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n}$$

We get so excited we call $A$ a different letter: $Q$

- if ~~Q is square~~, then We say $Q$ is an orthogonal matrix!!!
- in that case $Q^T = Q^{-1}$
- if $A$'s columns are not orthonormal, then we make them so using sneaky methods (actually brute force)

But what if our basis is not orthogonal? ☹

[Creating an orthogonal basis for a subspace is the harder step]

(aside) We will see we sometimes get them for free excitement!!!
V. important for $A\vec{x} = \vec{b}$ & SVD

Naturally, we use the "Gram-Schmidt method: process" //

(See * on L12 Ch 4)

Consider

basis for $R^3$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \& \vec{a}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Clearly $\vec{a}_1^T \cdot \vec{a}_2 = 2$ so $\vec{a}_1 \not\perp \vec{a}_2$.
We have work to do!

make $\vec{c}_1, \vec{c}_2, \& \vec{c}_3$ using these $\vec{a}_i$'s so the $\vec{c}$'s are all orthogonal

① $\vec{c}_1 = \vec{a}_1$

② $\vec{c}_2 = \vec{a}_2 - \frac{\vec{c}_1^T \vec{a}_2}{\vec{c}_1^T \vec{c}_1} \vec{c}_1$

projection of $\vec{a}_2$ onto $\vec{c}_1$

③ $\vec{c}_3 = \vec{a}_3 - \frac{\vec{c}_1^T \vec{a}_3}{\vec{c}_1^T \vec{c}_1} \vec{c}_1 - \frac{\vec{c}_2^T \vec{a}_3}{\vec{c}_2^T \vec{c}_2} \vec{c}_2$.

explain

proj of $\vec{a}_3$ onto $\vec{c}_1$

proj of $\vec{a}_3$ onto $\vec{c}_2$.

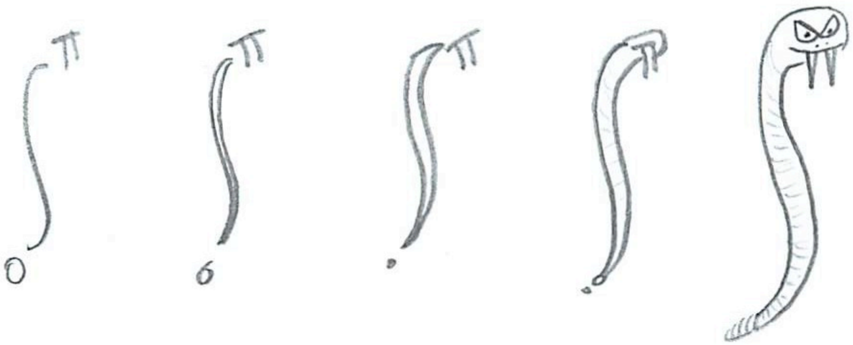
$$\vec{c}_1 = \vec{a}_1$$

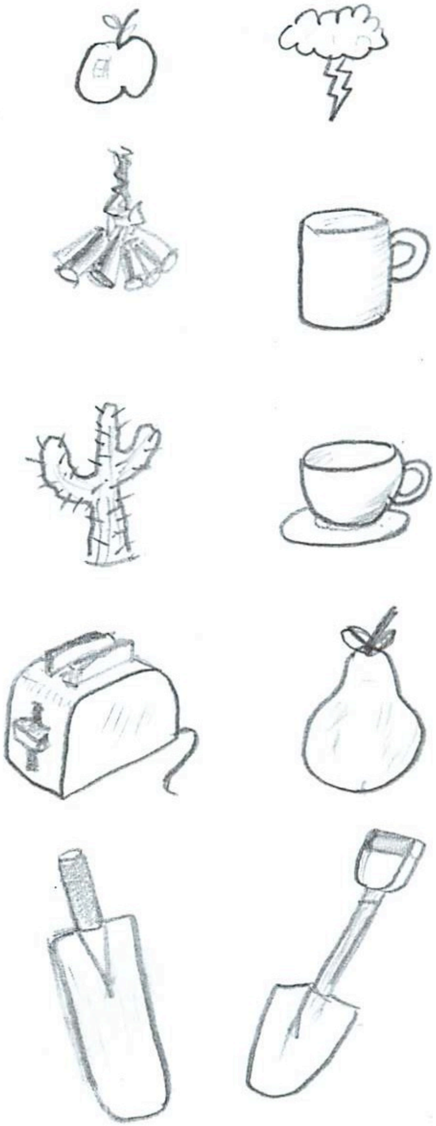
$$\vec{c}_2 = \vec{a}_2 - \left(\frac{\vec{c}_1^{\,T}\vec{c}_2}{\vec{c}_1^{\,T}\vec{c}_1}\right)\vec{c}_1 \Rightarrow \vec{a}_2 = (\hat{q}_1^T\vec{a}_2)\hat{q}_1 + \underbrace{\vec{c}_2}_{(\hat{q}_2^T \vec{a}_2)\hat{q}_2}$$

$$\left(\begin{aligned} \vec{a}_3 &= x_1\hat{q}_1 + x_2\hat{q}_2 + x_3\hat{q}_3 \\ x_i &= \hat{q}_i^T\vec{a}_3 \end{aligned}\right)$$

↖ do this!

$$\int_0^{\frac{\pi}{2}}$$





## LINEAR ALGEBRA

see ✱
on L12
ch4

$\vec{c}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\frac{2}{1}$

$$\vec{c}_2 = \begin{bmatrix}2\\0\\3\end{bmatrix} - \frac{[1\ 0\ 0]\begin{bmatrix}2\\0\\3\end{bmatrix}}{[1\ 0\ 0]\begin{bmatrix}1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\3\end{bmatrix}$$

makes sense!

$$\vec{c}_3 = \begin{bmatrix}4\\5\\6\end{bmatrix} - \frac{[1\ 0\ 0]\begin{bmatrix}4\\5\\6\end{bmatrix}}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{[0\ 0\ 3]\begin{bmatrix}4\\5\\6\end{bmatrix}}{[0\ 0\ 3]\begin{bmatrix}0\\0\\3\end{bmatrix}}\begin{bmatrix}0\\0\\3\end{bmatrix} = \begin{bmatrix}0\\5\\0\end{bmatrix}$$

4 $\frac{18}{9}$ 2 ✓.

Normalize

$$\hat{q}_i = \frac{\vec{c}_i}{\|\vec{c}_i\|} \qquad \hat{q}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \hat{q}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix},\ \hat{q}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Now

We like to use matrices to represent our manipulations

g $\mathbb{A} = \mathbb{L}\mathbb{U}$ = elimination.

What's the factorization here?

Go back to Gram Schmidt process & express $\vec{a}$'s in terms of the $\hat{q}$'s

projection matrices

✱ know

$\vec{a}_1$ is in direction of $\hat{q}_1$

$\vec{a}_2$ has components in direction $\hat{q}_1$ & $\hat{q}_2$

$\vec{a}_3$ " " " " $\hat{q}_1, \hat{q}_2, \& \hat{q}_3$

see Ch6 L2

Break into pieces by projection & sum up

$$\vec{a}_1 = \underbrace{\hat{q}_1\hat{q}_1^T}_{\text{projection matrix for a 1 d subspace}}\vec{a}_1$$

$\mathbb{P} = \mathbb{A}(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T$. $\mathbb{A} = \hat{q}_1$

improve this section!!!!!!

$$\vec{a}_2 = \hat{q}_1\hat{q}_1^T\vec{a}_2 + \hat{q}_2\hat{q}_2^T\vec{a}_2$$

Ch4 L11

$$\vec{a}_3 = \hat{q}_1\hat{q}_1^T\vec{a}_3 + \hat{q}_2\hat{q}_2^T\vec{a}_3 + \hat{q}_3\hat{q}_3^T\vec{a}_3$$

$$\underset{\mathbb{A}}{\begin{bmatrix}| & | & |\\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3\\ | & | & |\end{bmatrix}} = \underset{\mathbb{Q}}{\begin{bmatrix}| & | & |\\ \hat{q}_1 & \hat{q}_2 & \hat{q}_3\\ | & | & |\end{bmatrix}}\underset{\mathbb{R}}{\begin{bmatrix}\hat{q}_1^T\vec{a}_1 & \hat{q}_1^T\vec{a}_2 & \hat{q}_1^T\vec{a}_3\\ 0 & \hat{q}_2^T\vec{a}_2 & \hat{q}_2^T\vec{a}_3\\ 0 & 0 & \hat{q}_3^T\vec{a}_3\end{bmatrix}}$$

better way to do?

The 'QR' factorization of $\mathbb{A}$

n.b. only okay if $\mathbb{A}$'s columns are independent.
(no nullspace: n = r)

ex.

$$\begin{bmatrix}1 & 2 & 4\\0 & 0 & 5\\0 & 3 & 6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 4\\0 & 3 & 6\\0 & 0 & 5\end{bmatrix}$$

$\hat{q}_1\ \hat{q}_2\ \hat{q}_3$

actually a permutation matrix

$$\begin{bmatrix}1 & 2 & 3\\-1 & 0 & -3\\0 & -2 & 3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\sqrt{2} & \sqrt{2} & \sqrt{8}\\0 & \sqrt{6} & -\sqrt{6}\\0 & 0 & \sqrt{3}\end{bmatrix}$$

$\mathbb{R}$ is always upper triangular

BIG help with solving $\mathbb{A}\vec{x} = \vec{b}$

Normal eq: $\mathbb{A}^T\mathbb{A}\vec{x}^* = \mathbb{A}^T\vec{b}$

if $\mathbb{A} = \mathbb{Q}\mathbb{R}$,

$$\mathbb{A}^T\mathbb{A} = (\mathbb{Q}\mathbb{R})^T(\mathbb{Q}\mathbb{R}) = \mathbb{R}^T\overset{\mathbb{I}}{\cancel{\mathbb{Q}^T\mathbb{Q}}}\mathbb{R}$$
$$= \mathbb{R}^T\mathbb{R}$$

$$\mathbb{R}^T\mathbb{R}\vec{x}^* = \mathbb{R}^T\mathbb{Q}^T\vec{b}$$

invertible!!! upper triangular ◻, cols indep

$$\Rightarrow \boxed{\mathbb{R}\vec{x}^* = \mathbb{Q}^T\vec{b}}$$

solve by back substitution

Fast!!