Outline

Review for Exam 2

Words

Pictures
Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1–4.3)

Main pieces:

1. Big Picture of $A\vec{x} = \vec{b}$

2. Projections and the normal equation

As always, want ‘doing’ and ‘understanding’ abilities.
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- **Main pieces:**
  1. Big Picture of $A\vec{x} = \vec{b}$
  2. Projections and the normal equation
- As always, want ‘doing’ and ‘understanding’ abilities.
Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- **Main pieces:**
  1. Big Picture of $A\vec{x} = \vec{b}$
     - Must be able to draw the big picture!
  2. Projections and the normal equation
- **As always, want ‘doing’ and ‘understanding’ abilities.**
Stuff to know/understand

Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.
- Concept of a basis.
- Basis = minimal spanning set.
- Concept of orthogonal complement.
- Various techniques for finding bases and orthogonal complements.
Stuff to know/understand

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- Various techniques for finding bases and orthogonal complements.
Stuff to know/understand:

**Fundamental Theorem of Linear Algebra:**

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^T$.
- Column space $C(A) \subset \mathbb{R}^m$.
- Left Nullspace $N(A^T) \subset \mathbb{R}^m$.
- $\dim C(A) + \dim N(A^T) = r + (m - r) = m$
- Orthogonality: $C(A) \bigotimes N(A^T) = \mathbb{R}^m$
- Row space $C(A^T) \subset \mathbb{R}^n$.
- (Right) Nullspace $N(A) \subset \mathbb{R}^n$.
- $\dim C(A^T) + \dim N(A) = r + (n - r) = n$
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**Fundamental Theorem of Linear Algebra:**
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Stuff to know/understand:

**Fundamental Theorem of Linear Algebra:**

- Applies to any $m \times n$ matrix $\mathbf{A}$.
- Symmetry of $\mathbf{A}$ and $\mathbf{A}^T$.

- Column space $C(\mathbf{A}) \subset \mathbb{R}^m$.
- Left Nullspace $N(\mathbf{A}^T) \subset \mathbb{R}^m$.
- $\dim C(\mathbf{A}) + \dim N(\mathbf{A}^T) = r + (m - r) = m$
- Orthogonality: $C(\mathbf{A}) \otimes N(\mathbf{A}^T) = \mathbb{R}^m$

- Row space $C(\mathbf{A}^T) \subset \mathbb{R}^n$.
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Stuff to know/understand:

**Fundamental Theorem of Linear Algebra:**

- Applies to any $m \times n$ matrix $A$.
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- Column space $C(A) \subset R^m$.
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- $\dim C(A) + \dim N(A^T) = r + (m - r) = m$
- Orthogonality: $C(A) \otimes N(A^T) = R^m$
- Row space $C(A^T) \subset R^n$.
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Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $A$ to $\mathbb{R}_A$ and $A^T$ to $\mathbb{R}_{A^T}$.
- Understand crucial nature of $\mathbb{R}_A$ and $\mathbb{R}_{A^T}$.
- Identify pivot columns and free columns.
- Rank $r$ of $A = \#$ pivot columns.
- Know that relationship between $\mathbb{R}_A$’s columns hold for $A$’s columns.
Stuff to know/understand:

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Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $A$ to $R_A$ and $A^T$ to $R_{A^T}$.
- Understand crucial nature of $R_A$ and $R_{A^T}$.
- Identify pivot columns and free columns.
- Rank $r$ of $A = \#$ pivot columns.
- Know that relationship between $R_A$’s columns hold for $A$’s columns.
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Stuff to know/understand:

Bases for column space—three ways:

1. Find when $A\vec{x} = \vec{b}$ has a solution:
   - Reduce $[A | \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$'s elements for a solution to $A\vec{x} = \vec{b}$ to exist → obtain basis for $C(A)$.

2. Use $R_A$:
   - Find pivot columns in $R_A$—same columns in $A$ form a basis for $C(A)$.
   - Warning: $R_A$'s columns do not give a basis for $C(A)$

3. Use $R_{AT}$:
   - Best and easiest way: basis for column space = non-zero rows in $R_{AT}$, the reduced form of $A^T$.

Basis for row space:

- Take non-zero rows in $R_A$ (easy!).
- Matches way 3 for column space.
Stuff to know/understand:

Bases for column space—three ways:

1. Find when $A\vec{x} = \vec{b}$ has a solution:
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Bases for column space—three ways:

1. Find when \( A \tilde{x} = \tilde{b} \) has a solution:
   - Reduce \([A \mid \tilde{b}]\) where \( \tilde{b} \) is general.
   - Find conditions on \( \tilde{b}'s \) elements for a solution to \( A \tilde{x} = \tilde{b} \) to exist \( \rightarrow \) obtain basis for \( C(A) \).

2. Use \( R_A \):
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**Bases for column space—three ways:**

1. **Find when** $A\vec{x} = \vec{b}$ **has a solution:**
   - Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$’s elements for a solution to $A\vec{x} = \vec{b}$ to exist $\rightarrow$ obtain basis for $C(A)$.

2. **Use** $R_A$:
   - Find pivot columns in $R_A$—same columns in $A$ form a basis for $C(A)$.
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3. **Use** $R_{AT}$:
   - Best and easiest way: basis for column space = non-zero rows in $R_{AT}$, the reduced form of $A^T$.

**Basis for row space:**

- Take non-zero rows in $R_A$ (easy!).
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Stuff to know/understand:

## Bases for column space—three ways:

1. **Find when \( A\vec{x} = \vec{b} \) has a solution:**
   - Reduce \([A | \vec{b}]\) where \( \vec{b} \) is general.
   - Find conditions on \( \vec{b} \)'s elements for a solution to \( A\vec{x} = \vec{b} \) to exist → obtain basis for \( C(A) \).

2. **Use \( R_A \):**
   - Find pivot columns in \( R_A \)—same columns in \( A \) form a basis for \( C(A) \).
   - **Warning:** \( R_A \)'s columns do not give a basis for \( C(A) \).

3. **Use \( R_{AT} \):**
   - Best and easiest way: basis for column space = non-zero rows in \( R_{AT} \), the reduced form of \( A^T \).

## Basis for row space:

- Take non-zero rows in \( R_A \) (easy!).
- Matches way 3 for column space.
Stuff to know/understand:

Bases for column space—three ways:

1. Find when $\mathbf{Ax} = \mathbf{b}$ has a solution:
   - Reduce $[\mathbf{A} | \mathbf{b}]$ where $\mathbf{b}$ is general.
   - Find conditions on $\mathbf{b}$’s elements for a solution to $\mathbf{Ax} = \mathbf{b}$ to exist $\rightarrow$ obtain basis for $C(\mathbf{A})$.

2. Use $\mathbf{R}_\mathbf{A}$:
   - Find pivot columns in $\mathbf{R}_\mathbf{A}$—same columns in $\mathbf{A}$ form a basis for $C(\mathbf{A})$.
   - Warning: $\mathbf{R}_\mathbf{A}$’s columns do not give a basis for $C(\mathbf{A})$.

3. Use $\mathbf{R}_{\mathbf{AT}}$:
   - Best and easiest way: basis for column space = non-zero rows in $\mathbf{R}_{\mathbf{AT}}$, the reduced form of $\mathbf{A}^T$.

Basis for row space:

- Take non-zero rows in $\mathbf{R}_\mathbf{A}$ (easy!).
- Matches way 3 for column space.
Stuff to know/understand:

Bases for column space—three ways:

1. Find when $\mathbf{A}\vec{x} = \vec{b}$ has a solution:
   - Reduce $[\mathbf{A} | \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$’s elements for a solution to $\mathbf{A}\vec{x} = \vec{b}$ to exist → obtain basis for $C(\mathbf{A})$.

2. Use $\mathbf{R}_\mathbf{A}$:
   - Find pivot columns in $\mathbf{R}_\mathbf{A}$—same columns in $\mathbf{A}$ form a basis for $C(\mathbf{A})$.
   - Warning: $\mathbf{R}_\mathbf{A}$’s columns do not give a basis for $C(\mathbf{A})$.

3. Use $\mathbf{R}_{\mathbf{A}^T}$:
   - Best and easiest way: basis for column space = non-zero rows in $\mathbf{R}_{\mathbf{A}^T}$, the reduced form of $\mathbf{A}^T$.

Basis for row space:

- Take non-zero rows in $\mathbf{R}_\mathbf{A}$ (easy!).
- Matches way 3 for column space.
Stuff to know/understand:

Bases for column space—three ways:

1. Find when $A\vec{x} = \vec{b}$ has a solution:
   - Reduce $[A | \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$'s elements for a solution to $A\vec{x} = \vec{b}$ to exist $\rightarrow$ obtain basis for $C(A)$.

2. Use $R_A$:
   - Find pivot columns in $R_A$—same columns in $A$ form a basis for $C(A)$.
   - Warning: $R_A$'s columns do not give a basis for $C(A)$.

3. Use $R_{AT}$:
   - Best and easiest way: basis for column space = non-zero rows in $R_{AT}$, the reduced form of $A^T$.

Basis for row space:

- Take non-zero rows in $R_A$ (easy!).
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Stuff to know/understand:

**Bases for column space—three ways:**

1. **Find when** $Ax = b$ **has a solution:**
   - Reduce $[A \mid b]$ where $b$ is general.
   - Find conditions on $b$’s elements for a solution to $Ax = b$ to exist $\rightarrow$ obtain basis for $C(A)$.

2. **Use $R_A$:**
   - Find pivot columns in $R_A$—same columns in $A$ form a basis for $C(A)$.
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3. **Use $R_{AT}$:**
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**Basis for row space:**

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Stuff to know/understand:

Bases for column space—three ways:

1. Find when $\mathbf{A}\vec{x} = \vec{b}$ has a solution:
   - Reduce $[\mathbf{A} | \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$’s elements for a solution to $\mathbf{A}\vec{x} = \vec{b}$ to exist $\rightarrow$ obtain basis for $C(\mathbf{A})$.

2. Use $\mathbb{R}\mathbf{A}$:
   - Find pivot columns in $\mathbb{R}\mathbf{A}$—same columns in $\mathbf{A}$ form a basis for $C(\mathbf{A})$.
   - Warning: $\mathbb{R}\mathbf{A}$’s columns do not give a basis for $C(\mathbf{A})$.

3. Use $\mathbb{R}\mathbf{A}^T$:
   - Best and easiest way: basis for column space = non-zero rows in $\mathbb{R}\mathbf{A}^T$, the reduced form of $\mathbf{A}^T$.

Basis for row space:

- Take non-zero rows in $\mathbb{R}\mathbf{A}$ (easy!).
- Matches way 3 for column space.
Stuff to know/understand:

Bases for column space—three ways:

1. Find when $A\vec{x} = \vec{b}$ has a solution:
   - Reduce $[A | \vec{b}]$ where $\vec{b}$ is general.
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   - Find pivot columns in $R_A$—same columns in $A$ form a basis for $C(A)$.
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3. Use $R_{AT}$:
   - **Best and easiest way:** basis for column space = non-zero rows in $R_{AT}$, the reduced form of $A^T$.

Basis for row space:

- Take non-zero rows in $R_A$ (easy!).
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Stuff to know/understand:

Bases for column space—three ways:

1. Find when $A\vec{x} = \vec{b}$ has a solution:
   - Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$’s elements for a solution to $A\vec{x} = \vec{b}$ to exist $\rightarrow$ obtain basis for $C(A)$.

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   - Find pivot columns in $R_A$—same columns in $A$ form a basis for $C(A)$.
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3. Use $R_{A^T}$:
   - Best and easiest way: basis for column space = non-zero rows in $R_{A^T}$, the reduced form of $A^T$.

Basis for row space:

- Take non-zero rows in $R_A$ (easy!).
- Matches way 3 for column space.
Stuff to know/understand:

Bases for nullspaces, left and right:

- Basis for nullspace obtained by solving \( A \vec{x} = \vec{0} \)
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- \# free variables = \( n - \# \) pivot variables = \( n - r = \text{dim } N(A) \).
- Similarly find basis for \( N(A^T) \) by solving \( A^T \vec{y} = \vec{0} \).
- \( \text{dim } N(A^T) = m - r \).
- Key: Find bases for both nullspaces directly from \( \mathbb{R}_A \) and \( \mathbb{R}_{A^T} \).
Stuff to know/understand:

Bases for nullspaces, left and right:

- Basis for nullspace obtained by solving $\mathbf{A}\vec{x} = \vec{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- $\# \text{ free variables} = n - \# \text{ pivot variables} = n - r = \dim N(\mathbf{A})$.
- Similarly find basis for $N(\mathbf{A}^T)$ by solving $\mathbf{A}^T\vec{y} = \vec{0}$.
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- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- # free variables = $n - \#$ pivot variables = $n - r = \dim N(A)$.
- Similarly find basis for $N(A^T)$ by solving $A^T\vec{y} = \vec{0}$.
- $\dim N(A^T) = m - r$.
- Key: Find bases for both nullspaces directly from $\mathbb{R}_A$ and $\mathbb{R}_{A^T}$. 
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Number of solutions to $A\vec{x} = \vec{b}$:

1. If $\vec{b} \notin C(A)$, there are no solutions.
2. If $\vec{b} \in C(A)$, there is either one unique solution or infinitely many solutions.
   - Number of solutions now depends entirely on $N(A)$.
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- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b} = \vec{p} + \vec{e}$
- $\vec{p}$ = that part of $\vec{b}$ that lies in the line:

$$\vec{p} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \vec{a} \left( = \frac{\vec{a} \vec{a}^T}{\vec{a}^T \vec{a}} \vec{b} \right)$$

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- Understand construction and use of subspace projection operator $\mathbb{P}$:

$$\mathbb{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T,$$

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The symmetry of $A\vec{x} = \vec{b}$ and $A^T\vec{y} = \vec{c}$:

- **Null Space**
  - $A\vec{x}_r = \vec{b}$
  - $A^T\vec{y}_r = \vec{c}$
  - $A\vec{x}_n = \vec{0}$
  - $A^T\vec{y}_n = \vec{0}$

- **Column Space**
  - $C(A)$
  - $C(A^T)$

- **Row Space**
  - $R^n$
  - $R^m$

- **Left Null Space**
  - $N(A)$
  - $N(A^T)$

- Dimensions:
  - $d = r$
  - $d = n - r$
  - $d = m - r$
How $A\vec{x} = \vec{b}$ works:

- **Row Space** $A\vec{x}_r = \vec{b}$
- **Column Space** $A\vec{x} = \vec{b}$
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Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:

- $\vec{p}$ is in the column space of $A$.
- $\vec{e}$ is in the null space of $A$.
- $\vec{x}_* = \vec{x}_r + \vec{x}_n$.

$A\vec{x}_r = \vec{p}$

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The fourfold ways of $\mathbf{Ax} = \mathbf{b}$:

<table>
<thead>
<tr>
<th>case</th>
<th>example $R$</th>
<th>big picture</th>
<th># solutions</th>
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<td>$m = r$</td>
<td>$n = r$</td>
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<td>1 always</td>
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<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
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<tr>
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